Compressible Water Waves with Vorticity: An Overview

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Abstract

We survey some recent results concerning the motion of 3D water waves described by the compressible Euler equations in an unbounded domain with a moving top boundary and a fixed flat bottom. The water waves are influenced by gravity and surface tension, and the velocity is not assumed to be irrotational. These notes provide an overview of the recent results concerning compressible water waves. Specifically, we exhibit a newly developed unified approach through which we can prove the local well-posedness, incompressible limit, and zero surface tension limit in one attempt.

Contents

1 Introduction

We study the compressible Euler equations describing the motion of a barotropic liquid:

$$\begin{cases} \rho(\partial_t + u\cdot\nabla)u + \nabla p + \rho g e_3 = 0, & \text{in } \mathcal{D}, \\ \partial_t\rho + \nabla\cdot(\rho u) = 0, & \text{in } \mathcal{D}, \\ p = p(\rho), & \text{in } \mathcal{D}, \end{cases} \tag{1.1}$$

where $\mathcal{D} = \cup_{t\in[0,T]}\{t\}\times\mathcal{D}_t$ with $\mathcal{D}_t$ to be the moving domain filled with the liquid given by

$$\mathcal{D}_t := \{(x_1, x_2, x_3)\in\mathbb{R}^3 : -b < x_3 < \psi(t, x_1, x_2)\}, \quad b > 10.$$

In the sequel of these notes, we use $\Sigma_t = \{x_3 = \psi(t, x_1, x_2)\}$ to denote the moving surface boundary, while $\Sigma_b = \{x_3 = -b\}$ to denote the fixed flat bottom. Also, we will consider the case when $\Sigma_t \cap \Sigma_b = \emptyset$. This is easy to achieve in a short time interval by assuming $\|\psi(0,\cdot)\|_{L^\infty(\mathbb{R}^2)} < \frac{b}{2}$.

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In (1.1) above, we denote by u, ρ, p the velocity, density, and pressure, respectively. Meanwhile, the fluid is under influence of the gravity given by $\rho g e_3$, where $g > 0$ is the gravitational constant and $e_3 = (0,0,1)^T$. The equation of states $p = p(\rho)$ is assumed to be a strictly increasing function,

$$p'(\rho) > 0, \quad \text{whenever } \rho \geq \bar{\rho}_0,$$

which is required so that the system of equations (1.1) is closed. Here, $\bar{\rho}_0$ is a positive constant in the case when the fluid is a liquid, as opposed to a gas where the $\bar{\rho}_0 = 0$. For simplicity, we assume $\bar{\rho}_0 = 1$ in the sequel.

1.1 The initial and boundary conditions

Let $D_t = \partial_t + u \cdot \nabla$ be the material derivative. For the free-boundary problem (1.1) to be solvable, we impose the initial data

$$\mathcal{D}_t|_{t=0} = \mathcal{D}_0, \quad u|_{t=0} = u_0, \quad \rho|_{t=0} = \rho_0, \tag{1.2}$$

as well as the boundary conditions

$$D_t|_{\Sigma_t} \in T(\Sigma_t), \quad p|_{\Sigma_t} = \sigma\mathcal{H}, \quad \text{and } u^3|_{\Sigma_b} = 0. \tag{1.3}$$

In the first boundary condition, $T(\Sigma_t)$ is the *tangent bundle* of Σ_t, and $D_t|_{\Sigma_t} \in T(\Sigma_t)$ is the so-called kinematic boundary condition, which states that the free interface moves with the normal component of the velocity (See the remark after Notation 3.1). The second boundary condition states that the pressure agrees with the surface tension on the moving boundary, where $\sigma > 0$ is the *surface tension coefficient*, and $\mathcal{H}$ is the mean curvature. Finally, the third boundary condition is the standard slip boundary condition imposed on the fixed flat bottom.

1.2 The equation of states, Mach number, and Rayleigh-Taylor sign condition

One infers from the definition that the equation of states $p = p(\rho)$ provides a one-to-one correspondence between the pressure and density of the fluid. Physically, the sound speed of the fluid is given by

$$c_s(\rho) := \sqrt{p'(\rho)}.$$

It is known that "the stiffness" or "the incompressibility" of the fluid is measured by c_s. Particularly, the fluid becomes incompressible if $c_s = +\infty$.

We are interested in the behavior of the water waves as c_s tends to $+\infty$. Mathematically, we can simplify this procedure by viewing the sound speed as a *family of parameters*. As in [9, 10, 11, 30, 31], we consider a family of pressures $\{p_{\lambda'}(\rho)\}$ parametrized by $\lambda' \in (0, +\infty)$ satisfying $\lambda' := \sqrt{p'_{\lambda'}(\rho)}|_{\rho=1}$. From now on, we shall refer to the number λ' as the sound speed after a slight abuse of terminology.

A standard choice of the equation of states would be

$$p_{\lambda'}(\rho) = \frac{1}{\gamma}(\lambda')^2(\rho^\gamma - 1), \quad \text{for some fixed } \gamma \geq 1.$$

On the other hand, let $\mathcal{F}_\lambda(p) := \log(\rho_\lambda(p))$ with $\lambda = (\lambda')^{-1}$. Then

$$\log(\rho_\lambda(p)) = \frac{1}{\gamma}\log\left(\gamma\lambda^2 p + 1\right).$$

Since the Mach number $= u c_s^{-1}$ in physics, and because we expect that our velocity u remains bounded[1] in the interval of existence, we shall again slightly abuse the terminology and call λ the Mach number.

On the other hand, when the surface tension is absent (i.e., $\sigma = 0$), the compressible water waves (1.1), (1.2)–(1.3) is known to be *ill-posed* [12] unless the Rayleigh-Taylor sign condition

$$-\nabla_N p \geq c > 0, \quad N = (-\partial_1\psi, -\partial_2\psi, 1)^T \tag{1.4}$$

holds. In other words, the Rayleigh-Taylor sign condition has to be imposed while taking the zero surface tension limit. Since we are taking the localized initial data (i.e., u decays to 0 as $|x| \to +\infty$), the presence of the gravity is essential to ensure that (1.4) remains valid everywhere on Σ_t. This can be seen by taking the normal component of the momentum equation $\rho(\partial_t + u \cdot \nabla)u = -(\nabla p + \rho g e_3)$.

Moreover, we mention here that (1.4) will be modified slightly to fit the energy estimate in Section 4 because the moving boundary is a graph (see (4.3)).

[1] This is indeed the case since the existence theory shows that $u \in H^4$, and so $u \in C^{2,\frac{1}{2}}$.

1.3 Some existing related results concerning water waves

The study of water waves has blossomed over the past three decades or so. Most of the previous studies focused on the *incompressible* and *irrotational* water waves[2], where the *incompressible* velocity w satisfies $\nabla \cdot w = 0$ and $\nabla \times w = \mathbf{0}$, and the density $\rho = 1$. The irrotational assumption is a condition that is preserved by evolution by studying its evolution equation obtained by taking the curl operator to the momentum equation. Also, it is important to mention that the fluid pressure p is no longer determined by the equation of states but appears as a Lagrangian multiplier enforcing the divergence-free constraint.

The first breakthrough came in Wu [43, 44], where the LWP for both 2D and 3D incompressible water waves is established assuming $\nabla \times w = \mathbf{0}$. Meanwhile, the incompressible and irrotational water wave problem has attracted great attention for the long-time existence of small solutions. Previous works mostly focused on the case of an unbounded domain diffeomorphic to either the lower half-space (i.e., infinite depth) or the strip $\mathbb{R}^{d-1} \times (-b, 0)$ ($d = 2, 3$), and we refer to Wu [45, 46] for the first breakthrough, and numerous related works [13, 14, 2, 24, 8, 19, 18, 20, 21, 42, 48]. See also [22, 39] for some special cases when the vorticity is nonzero.

On the other hand, the theory concerns compressible water waves are much less well-developed. Most previous works study the motion of a compressible *liquid* with free interface focus on the case of *bounded domains*. We refer to Lindblad [28, 29] for the first existence result using the Nash-Moser iteration, and other related works [7, 15] for LWP using the classical energy method. In the compressible gravity water wave problem without surface tension, Trakhinin [40] was the first to prove the LWP via the Nash-Moser iteration which leads to a loss of regularity from initial data to solution. In [33], the author and Zhang proved the LWP for the compressible gravity water waves using the classical energy approach. Very recently, we proved the LWP for the compressible gravity-capillary water waves in [32].

1.4 Some existing results concerning the incompressible and zero surface tension limits

The incompressible limit for Euler equations has been studied extensively in absence of the moving surface boundary, see, e.g., [25, 26, 11, 38, 41, 4, 23, 35, 1, 9]. However, much less is known about the incompressible limit of free-surface inviscid fluids. The first result was due to the author's paper with Lindblad [30] for the case of a bounded domain and zero surface tension. Later, the author extended the result of [30] to study the compressible gravity water waves [31], which appears to be the first result that concerns the incompressible limit of the gravity water waves. Also, in [10], Disconzi and the author proved the incompressible limit for the free-boundary Euler equations in a bounded fluid domain with surface tension.

We remark here that although we can obtain a solution for the compressible gravity water waves through the energy proposed in [33], it is not uniform in the Mach number and thus we cannot derive the incompressible limit while constructing the solution. Moreover, the energy constructed in [31] is based on the Christodoulou-Lindblad Q-tensor introduced in [6] and it turns out to be difficult to use this energy to prove the local existence. Recently, Zhang [47] extends the method of [33] to study the compressible elastodynamics and obtain the local well-posedness and incompressible limit together. However, we find that it is difficult to adapt the methodology of [47] to the case with surface tension[3].

As for the zero surface tension problem, Ambrose and Masmoudi [3] considered the irrotational 2D incompressible water waves. See also [36] for incompressible water waves in general spatial dimensions. As for the free-boundary compressible Euler equations, Coutand, Hole, and Shkoller [7] proved the LWP and the zero surface tension limit in a bounded domain.

1.5 Main objectives

We survey the results in recent papers by the author and J. Zhang [32] and [33] but with a focus on the former. Particularly, we prove:

a. The local well-posedness (LWP) of the motion of compressible gravity-capillary water waves with vorticity. Given *sufficiently smooth* initial data, we show that there exists a $T > 0$, depending on the data, such that (1.1) equipped with (1.2) and (1.3) admits a unique solution $(\psi(t), u(t), \rho(t))$, $\forall t \in [0, T]$.
b. $(\psi(t), u(t), \rho(t))$ converges to the solution of the *incompressible gravity water waves with vorticity* when both the Mach number λ and surface tension coefficient σ tend to 0, provided that the Rayleigh-Taylor sign condition holds.

[2]When the velocity w is divergence and curl-free, its potential, denoted by ξ, is a harmonic function. Then this allows one to reduce the incompressible free-boundary Euler equations to a system of equations on the moving interface. In other words, one can reduce the original system of equations defined on a strip to a new system defined on Σ_t, which is close to $\mathbb{R}^2$ if ψ is small. However, no parallel formulation of this kind exists when studying compressible water waves with vorticity.

[3]Zhang [47] studied the free-boundary elastodynamics using Lagrangian coordinates. We shall explain in Subsection 2.3 on the difficulties when surface tension is taken into account.

We achieve these objectives by introducing a carefully designed approximate system of equations that are asymptotically consistent with the energy estimate. Specifically, we show that this energy estimate is uniform in λ, as well as in σ with the Rayleigh-Taylor sign condition. We refer to Theorems 4.2–4.3 for the precise statements related to the aforementioned objectives. Moreover, it is not hard to see that (b.) implies:

b.1. (The incompressible limit) $(\psi(t), u(t), \rho(t))$ converges to the solution of the *incompressible gravity-capillary water waves with vorticity* when $\lambda \to 0$, while keeping $\sigma > 0$ fixed.

b.2. (The zero surface tension limit) $(\psi(t), u(t), \rho(t))$ converges to the solution of the *compressible gravity water waves with vorticity* when $\sigma \to 0$, while keeping $\lambda > 0$ fixed, provided that the Rayleigh-Taylor sign condition holds.

Remark. It can be seen from (b.2) that we can obtain the LWP proved in [33] from the LWP for the gravity-capillary water waves in [32] by taking $\sigma \to 0$. Nevertheless, the approach we have in [33] is different compared to that in [32]. This will be explained in the upcoming section.

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2 Reformulation in Lagrangian coordinates

We convert the free-boundary compressible Euler equations to a system of equations defined on the fixed domain

$$\Omega := \{(x_1, x_2, x_3) : -b < x_3 < 0\},$$

whose boundaries are $\Sigma := \{x_3 = 0\}$ and $\Sigma_b := \{x_3 = -b\}$, respectively.

2.1 The Lagrangian coordinates

A commonly used method here is to employ the Lagrangian coordinates, which are characterized by the flow map of the velocity. Denoting by $\eta : [0, T] \times \Omega \to \mathcal{D}$ be the flow map, then η satisfies

$$\partial_t \eta(t, x) = u \circ \eta(t, x), \quad \eta(0, x) = \eta_0,$$

where $\eta_0 : \Omega \to \mathcal{D}_0$ is a given diffeomorphism. Let the cofactor matrix be $a = (\partial\eta)^{-1}$, specifically with $a^{\mu\alpha} := \frac{\partial x^\mu}{\partial \eta^\alpha}$, and $a^\mu_\alpha = \delta_{\nu\alpha} a^{\mu\nu}$. This is well-defined in the time interval $[0, T]$ when T is sufficiently small. Let f be a smooth function defined in $\mathcal{D}$. In these new coordinates, it can be seen that

$$D_t f = \partial_t (f \circ \eta), \quad \nabla_\alpha f := \frac{\partial f}{\partial \eta_\alpha} = a^\mu_\alpha \frac{\partial (f \circ \eta)}{\partial x_\mu}.$$

In other words, although D_t is reduced to ∂_t, new nonlinearities are introduced through the spatial derivative ∇.

2.2 Methodology of [33]: A brief review

We proved the LWP for the compressible gravity water waves with infinite depth (i.e., $\sigma = 0$ and $b = +\infty$) in [33] by expressing (1.1), (1.2) and (1.3) in $[0, T] \times \mathbb{R}^3_-(= \{x \in \mathbb{R}^3 : x_3 \leq 0\})$ via the Lagrangian coordinates. By introducing the enthalpy $h = h(\rho)$ defined by

$$h(\rho) = \int_1^\rho p'(l) l^{-1}\, dl,$$

the first two equations in (1.1) become

$$\begin{aligned} \partial_t (u \circ \eta)_\alpha + (\nabla_a)_\alpha (h \circ \eta) &= -g e_3, \\ (\nabla_a) \cdot (u \circ \eta) &= -\partial_t \log \rho(h \circ \eta), \end{aligned} \tag{2.1}$$

where $(\nabla_a)_\alpha := a^\mu_\alpha \partial_{x_\mu}$, with the boundary condition

$$h \circ \eta = 0, \quad \text{on } \Sigma. \tag{2.2}$$

It is well-known that the a priori energy estimate of (2.1)–(2.2) does not lead to LWP directly, since it cannot be carried over to study the linearized equations as the structure of original equations is destroyed by linearization on Σ. To overcome this difficulty, we need to design a *approximate system of equations* indexed by $\kappa > 0$ that is asymptotically consistent with (2.1)–(2.2), which admits an *approximate solution* $(\eta_\kappa, u \circ \eta_\kappa, h \circ \eta_\kappa)$ that converges to the solution of (2.1)–(2.2) in H^s (for some large $s > 0$, say, e.g., $s = 4$.) as $\kappa \to +\infty$. In [33], such approximate equations are constructed via smoothing the nonlinear cofactor matrices in the tangential direction (i.e., the (x_1, x_2)-direction, see Definition 5.1 for details), and introducing a correction term ϕ on the RHS of the transport equation of η, which then becomes $\partial_t \eta = u \circ \eta + \phi$. This correction term[4] ϕ is utilized to kill the top-order boundary terms in the energy estimate generated by the tangential smoothing. We refer to [33, Section 2] for the detailed construction of ϕ.

2.3 Some Major difficulties when studying compressible water waves with surface tension using Lagrangian coordinates

However, it appears that it is difficult to use the Lagrangian coordinates to prove the LWP for the compressible water waves when $\sigma > 0$. Let $g_{ij} = \partial_{x_i}\eta \cdot \partial_{x_j}\eta$ with $1 \le i, j \le 2$ to be the induced metric on Σ, and Δ_g is the Laplacian associated with g. The boundary condition $p = \sigma\mathcal{H}$ becomes

$$p \circ \eta|_\Sigma = -\sigma \Delta_g \eta \cdot n, \tag{2.3}$$

where $n := \frac{a^T e_3}{|a^T e_3|}$ is the Eulerian unit normal of the moving surface boundary expressed in Lagrangian coordinates. Moreover, let $J = det(\partial\eta)$ and $A = Ja$. Because of $A^{3\alpha} n_\alpha = \sqrt{|g|}$, where $|g| = |\det g|$, we may re-express (2.3) as

$$A^{3\alpha}(p \circ \eta) = -\sigma \sqrt{|g|}(\Delta_g \eta \cdot n) n^\alpha, \tag{2.4}$$

which is easier to apply in the energy estimate. Now, we need the Hodge-type div-curl estimate (see, e.g., [17, Lemma 2.7], and the reference therein) to obtain the control of $\partial_t^k(u \circ \eta) \in H^{4-k}(\Omega)$, $k = 0, 1, 2, 3, 4$.

Lemma 2.1. Let X be a (smooth) vector field defined on Ω. Denote by $\overline{\partial} = (\partial_{x_1}, \partial_{x_2})$ the tangential derivative, it holds that

$$\|X\|^2_{H^s(\Omega)} \le C(\|\overline{\partial}\eta\|_{W^{1,\infty}(\Omega)}, \|\eta\|^2_{H^s(\Omega)}) \left(\|X\|^2_{L^2(\Omega)} + \|\nabla_a \cdot X\|^2_{H^{s-1}(\Omega)} + \|\nabla_a \times X\|^2_{H^{s-1}(\Omega)} + \|\overline{\partial}^s X\|^2_{L^2(\Omega)} \right). \tag{2.5}$$

Let $X = \partial_t^k(u \circ \eta)$ and $s = 4 - k$. Then we infer from (2.5) that the control of $\|\partial_t^k(u \circ \eta)\|_{H^{4-k}(\Omega)}$ requires that of $\|\overline{\partial}^{4-k}\partial_t^k(u \circ \eta)\|_{L^2(\Omega)}$. Let us consider only the case when $k = 4$, i.e., full-time derivatives. Taking ∂_t^4 to the momentum equation[5]

$$(\rho \circ \eta)\partial_t(u \circ \eta)_\alpha + (\nabla_a)_\alpha(p \circ \eta) = -(\rho \circ \eta) g e_3,$$

and testing it with $\partial_t^4(u \circ \eta)$ in $L^2(\Omega)$, (2.4) yields the appearance of the following boundary integral:

$$I = \sigma \int_\Sigma \partial_t^4(u \circ \eta)_\alpha \partial_t^4(\sqrt{|g|}\Delta_g \eta \cdot n n^\alpha)\, \mathrm{d}S.$$

Invoking the identity

$$\Delta_g \eta^\alpha = g^{ij}\partial_{x_i}\partial_{x_j}\eta \cdot n n^\alpha,$$

we obtain

$$I \stackrel{L}{=} \sigma \int_\Sigma \sqrt{|g|} g^{ij} \partial_t^4(u \circ \eta)_\alpha n^\alpha \partial_t^4(\partial_{x_i}\partial_{x_j}\eta \cdot n)\, \mathrm{d}S = \sigma \int_\Sigma \sqrt{|g|} g^{ij} [(\partial_t^4(u \circ \eta)) \cdot n][\partial_t^3(\partial_{x_i}\partial_{x_j}(u \circ \eta) \cdot n)]\, \mathrm{d}S.$$

[4]This correction term is not required when the surface tension is present.

[5]Note that we do not use the enthalpy formulation (2.1) here since it is easier (and more natural) to impose the surface tension on pressure.

Notation 2.2. Here and throughout, $A \stackrel{L}{=} B$ means that $A = B$ modulo some easy-to-control lower-order terms or minor error terms.

Therefore, after integrating ∂_{x_i} by parts in the last integral above, we find that I contributes to the energy term

$$-\frac{d}{dt}\frac{\sigma}{2}\int_\Sigma \sqrt{|g|}g^{ij}[\partial_{x_i}\partial_t^3((u\circ\eta)\cdot n)][\partial_{x_j}\partial_t^3((u\circ\eta)\cdot n)]\,\mathrm{d}S \approx -\frac{d}{dt}\frac{\sigma}{2}|\partial_t^3(u\circ\eta)\cdot n|^2_{H^1(\Sigma)}. \tag{2.6}$$

Nevertheless, there are some major error terms generated while producing (2.6), e.g.,

$$\pm\sigma\int_\Sigma \sqrt{|g|}g^{ij}[(u\circ\eta)\cdot\partial_t^4 n][\partial_t^3(\partial_{x_j}(u\circ\eta)\cdot\partial_{x_i}n)]\,\mathrm{d}S. \tag{2.7}$$

Since $\partial_t^4 n$ gives $\overline{\partial}\partial_t^3(u\circ\eta)$ at the leading order, this term requires us the control of $|\partial_t^3(u\circ\eta)|^2_{H^1(\Sigma)}$, which is **not** the energy term on the RHS of (2.6). This is possible in the incompressible case as one can boost the interior regularity of $\partial_t^k(u\circ\eta)$, $k = 0,1,2,3$ by a half derivatives (see [16, Section 3] for details). However, this no longer holds in the compressible case, and one has to invoke a dedicated cancellation scheme to achieve control instead. On the other hand, this cancellation scheme is destroyed when treating the approximate equations, since the Eulerian normal n in (2.4) will be replaced by the tangentially smoothed version. Fortunately, we find out in [32] that this is no longer an issue by fixing $\mathcal{D}_t$ using the coordinates characterized by the moving boundary as a graph $x_3 = \psi(t,x_1,x_2)$. This shall be discussed in the following section[6].

3 Reformulation in the coordinates characterized by $\psi(t, x_1, x_2)$

Recall that we assume the free interface $\Sigma_t = \{x_3 = \psi(t,x_1,x_2)\}$ does not touch the fixed flat bottom $\Sigma_b = \{x_3 = -b\}$ within $[0,T]$, we may assume $b > 10$ and $\|\psi_0\|_{L^\infty(\mathbb{R}^2)} \leq 1$ for simplicity. We consider a family of diffeomorphisms $\Phi(t,\cdot) : \Omega \to \mathcal{D}_t$ given by

$$\Phi(t,x_1,x_2,x_3) = (t,x_1,x_2,\varphi(t,x_1,x_2,x_3)),$$

where $\varphi(t,x_1,x_2,x_3) = x_3 + \chi(x_3)\psi(t,x_1,x_2)$. Here, $\chi \in C_c^\infty(-b,0]$ is a smooth cut-off function satisfying

$$\|\chi'\|_{L^\infty(-b,0]} \leq \frac{1}{\|\psi_0\|_{L^\infty(\mathbb{R}^2)}+1}, \quad \chi = 1 \text{ on } (-\delta_0, 0],$$

for some $\delta_0 > 0$ chosen sufficiently small. Throughout the rest of these notes, we set

$$x' = (x_1,x_2), \quad \text{and } x = (x',x_3),$$

$$\partial_j = \frac{\partial}{\partial x_j}, \quad j = 1,2,3,$$

and define

$$v(t,x) = u\circ\Phi(t,x), \quad \rho(t,x) = \mathrm{\rho}\circ\Phi(t,x), \quad q(t,x) = p\circ\Phi(t,x)$$

to be the velocity, density, and pressure, respectively, in $[0,T]\times\Omega$. Also, for any C^1-function $g(t,x)$, we have

$$\partial_\alpha g\circ\Phi = \partial_\alpha^\varphi(g\circ\Phi), \quad \alpha = t,1,2,3$$

with

$$\partial_\beta^\varphi = \partial_\beta - (\partial_\beta\varphi)(\partial_3\varphi)^{-1}\partial_3, \quad \beta = t,1,2, \quad \partial_3^\varphi = (\partial_3\varphi)^{-1}\partial_3.$$

In other words, we have

$$\begin{pmatrix}\partial_1^\varphi\\ \partial_2^\varphi\\ \partial_3^\varphi\end{pmatrix} = \begin{pmatrix}1 & 0 & -(\partial_1\varphi)(\partial_3\varphi)^{-1}\\ 0 & 1 & -(\partial_2\varphi)(\partial_3\varphi)^{-1}\\ 0 & 0 & (\partial_3\varphi)^{-1}\end{pmatrix}\begin{pmatrix}\partial_1\\ \partial_2\\ \partial_3\end{pmatrix}.$$

[6]Alternatively, Coutand, Hole, and Shkoller [7] proved the LWP for free-boundary compressible Euler equations with $\sigma > 0$ in a *bounded fluid domain* using parabolic regularization. We however find this method difficult to apply in the case of an *unbounded fluid domain*, especially when proving the existence alongside incompressible and zero surface tension limits.

where

$$\mathcal{A} := \begin{pmatrix} 1 & 0 & -(\partial_1\varphi)(\partial_3\varphi)^{-1} \\ 0 & 1 & -(\partial_2\varphi)(\partial_3\varphi)^{-1} \\ 0 & 0 & (\partial_3\varphi)^{-1} \end{pmatrix}^T \tag{3.1}$$

is the cofactor matrix. Thus, we see that, for each fixed $t \in [0, T]$, the change of variables $(x', x_3) \to (x', \varphi(t, x'))$ is well-defined as long as $\det\mathcal{A} = (\partial_3\varphi)^{-1} > 0$, which is indeed the case as $\partial_3\varphi(t, x', x_3) = 1 + \chi'(x_3)\psi(t, x')$ and

$$|\chi'(x_3)\psi(t, x')| \leq \frac{\|\psi(t, \cdot)\|_{L^\infty(\mathbb{R}^2)}}{\|\psi_0\|_{L^\infty(\mathbb{R}^2)} + 1} < 1,$$

if T is sufficiently small.

Notation 3.1. [Derivatives] We adopt the following notations in the rest of these notes.

i. (Flat space-time derivatives) ∂_t, and $\partial_j = \frac{\partial}{\partial x_j}$, $j = 1, 2, 3$.
ii. (The tangential derivatives) $\overline{\nabla} = \overline{\partial} = (\partial_1, \partial_2)$. In particular, $\overline{\partial}$ is tangent to both Σ and Σ_b.
iii. (Eulerian time derivative) $\partial_t^\varphi = \partial_t - (\partial_t\varphi)(\partial_3\varphi)^{-1}\partial_3$.
iv. (Eulerian spatial derivatives) $\nabla_\tau^\varphi = \partial_\tau^\varphi = \partial_\tau - (\partial_\tau\varphi)(\partial_3\varphi)^{-1}\partial_3$, $\tau = 1, 2$, and $\nabla_3^\varphi = \partial_3^\varphi = (\partial_3\varphi)^{-1}\partial_3$.
v. (The material derivative) $D_t^\varphi = \partial_t^\varphi + v\cdot\nabla^\varphi = \partial_t + \overline{v}\cdot\overline{\partial} + (\partial_3\varphi)^{-1}(v\cdot\mathbf{N} - \partial_t\varphi)\partial_3$, where $\overline{v} = (v^1, v^2)^T$, and $\mathbf{N} = (-\partial_1\varphi, \partial_2\varphi, 1)^T$. The second equality follows from the definitions of ∂_t^φ and ∂_j^φ, $j = 1, 2, 3$, listed above.

Remark. Note that $(v \cdot \mathbf{N} - \partial_t\varphi)|_\Sigma = (v \cdot N - \partial_t\psi) = 0$ because of the kinematic boundary condition, and so

$$D_t^\varphi|_\Sigma = (\partial_t + \overline{v} \cdot \overline{\partial}) \in T(\Sigma).$$

Also, since $v \cdot e_3 = 0$ on Σ_b, we have $(v \cdot \mathbf{N} - \partial_t\varphi)|_{\Sigma_b} = 0$. So

$$D_t^\varphi|_{\Sigma_b} = (\partial_t + \overline{v} \cdot \overline{\partial}) \in T(\Sigma_b).$$

Since the kinematic condition $D_t|_{\Sigma_t} \in T(\Sigma_t)$ is equivalent to $\partial_t\psi = v \cdot N$, where $N = (-\partial_1\psi, -\partial_2\psi, 1)^T$, and

$$\mathcal{H} = \overline{\nabla} \cdot \frac{N}{|N|} = -\overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1 + |\overline{\nabla}\psi|^2}}\right),$$

so (1.1), (1.2)–(1.3) is reduced to

$$\begin{cases} \rho D_t^\varphi v + \nabla^\varphi q = -\rho g e_3, & \text{in } [0, T] \times \Omega, \\ D_t^\varphi \rho + \rho\nabla^\varphi \cdot v = 0, & \text{in } [0, T] \times \Omega, \\ q = q(\rho), & \text{in } [0, T] \times \Omega, \\ \partial_t\psi = v \cdot N, & \text{on } [0, T] \times \Sigma, \\ q = -\sigma\overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}}\right), & \text{on } [0, T] \times \Sigma, \\ v^3 = 0, & \text{on } [0, T] \times \Sigma, \\ (\psi, v, \rho)|_{t=0} = (\psi_0, v_0, \rho_0). \end{cases} \tag{3.2}$$

Remark. The continuity equation, i.e., the second equation in (3.2) can also be expressed as

$$\partial_t^\varphi \rho + \nabla^\varphi \cdot (\rho v) = 0, \quad \text{in } [0, T] \times \Omega. \tag{3.3}$$

Also, let $\mathcal{F}(q) := \log\rho(q)$. Since $q'(\rho) > 0$, one must also have $\mathcal{F}'(q) > 0$. In particular, from the equation of states $q_\lambda(\rho) = \frac{1}{\gamma}(\lambda)^{-2}(\rho^\gamma - 1)$, we infer

$$\mathcal{F}_\lambda(q) = \frac{1}{\gamma}\log\left(\gamma\lambda^2 q + 1\right).$$

It can be seen that

$$\frac{\lambda^2}{C} \le \mathcal{F}_\lambda'(q) \le C\lambda^2$$

holds for some $C > 0$, i.e., $\mathcal{F}_\lambda' \approx \lambda^2$. Thanks to this, one may assume that

$$\mathcal{F}_\lambda'(q) = \lambda^2, \tag{3.4}$$

from now on for the sake of simple exposure. In fact, the general case is not hard to treat by imposing the assumption

$$|\mathcal{F}_\lambda^{(k)}(q)| \le C, \quad |\mathcal{F}_\lambda^{(k)}(q)| \le C|\mathcal{F}_\lambda'(q)|^k \le C\mathcal{F}_\lambda'(q), \quad 0 \le k \le 4,$$

under which the terms generated by differentiating $\mathcal{F}_\lambda'(q)$ contribute to only trivial lower-order errors.

The continuity equation $D_t^\varphi \rho + \rho \nabla^\varphi \cdot v = 0$ turns into $\mathcal{F}'(q) D_t^\varphi q + \nabla^\varphi \cdot v = 0$. Invoking the assumption (3.4), this becomes

$$\lambda^2 D_t^\varphi q + \nabla^\varphi \cdot v = 0. \tag{3.5}$$

3.1 Rewriting (3.2) with the modified pressure

Since $\rho \ge 1$ and $g > 0$, the RHS of the momentum equation $\rho D_t^\varphi v + \nabla^\varphi q = -\rho g e_3$ is not $L^2(\Omega)$-integrable. To resolve this issue, we define

$$\check{q} = q + g\varphi$$

to be the modified pressure, which satisfies $\nabla^\varphi \check{q} = \nabla^\varphi q + g e_3$ as $\nabla_j^\varphi \varphi = \delta_{3j}$. Also, (3.3) yields that $\rho - 1 \in L^2(\Omega)$ as long as $\rho_0 - 1 \in L^2(\Omega)$. Thus, the momentum equation becomes

$$\rho D_t^\varphi v + \nabla^\varphi \check{q} = -(\rho - 1) g e_3,$$

and all of its terms can be estimated in $L^2(\Omega)$. Also, because $D_t^\varphi \varphi = (\overline{v} \cdot \overline{\partial})\varphi + v \cdot \mathbf{N} = v^3$, (3.5) turns further into

$$\lambda^2 D_t^\varphi \check{q} + \nabla^\varphi \cdot v = g\lambda^2 v^3.$$

Therefore, from now on we shall study the modified compressible Euler equations expressed in $\psi, v, \rho, \check{q}$:

$$\begin{cases} \rho D_t^\varphi v + \nabla^\varphi \check{q} = -(\rho - 1) g e_3, & \text{in } [0,T] \times \Omega, \\ \lambda^2 D_t^\varphi \check{q} + \nabla^\varphi \cdot v = g\lambda^2 v^3, & \text{in } [0,T] \times \Omega, \\ \partial_t \psi = v \cdot N, & \text{on } [0,T] \times \Sigma, \\ \check{q} = -\sigma \overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}} \right) + g\psi, & \text{on } [0,T] \times \Sigma, \\ v^3 = 0, & \text{on } [0,T] \times \Sigma, \\ (\psi, v, \check{q})|_{t=0} = (\psi_0, v_0, \check{q}_0). \end{cases} \tag{3.6}$$

We can of course obtain the initial data $\check{q}_0$ from ρ_0 and φ_0, where the latter is determined by ψ_0.

4 The main results

The rest of these notes is devoted to surveying the main results of [32] and the key ideas with partial details.

Notation 4.1 (Sobolev norms)**.** We adopt the following notations in the sequel.

i. ($H^s(\Omega)$-norm) $\|\cdot\|_s = \|\cdot\|_{H^s(\Omega)}$.
ii. ($H^s(\Sigma)$-norm) $|\cdot|_s = \|\cdot\|_{H^s(\Sigma)}$.
iii. (L^∞-norms) $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(\Omega)}$, $|\cdot|_\infty = \|\cdot\|_{L^\infty(\Sigma)}$.

Theorem 4.2 (Local well-posedness)**.** Let $b > 10$, and the surface tension coefficient $\sigma > 0$ be fixed. Let $(\psi_0, v_0, \rho_0 - 1) \in H^5(\Sigma) \times H^4(\Omega) \times H^4(\Omega)$ be the initial data of (3.6) satisfying certain compatibility conditions, $|\psi_0|_\infty \le 1$, and $v_0^3|_{\Sigma_b} = 0$. Then there exists $T > 0$, depending only on the initial data, such that (3.6) admits a unique solution $(\psi(t), v(t), \rho(t))$ verifying

$$\sup_{t\in[0,T]} E(t) \le C(\sigma^{-1}, E(0)),$$

where

$$E(t) = \sum_{k=0}^{4} \left(\|\partial_t^k v(t)\|_{4-k}^2 + \sigma |\overline{\nabla} \partial_t^k \psi(t)|_{4-k}^2 \right) + \left(\|\partial \check{q}(t)\|_3^2 + \sum_{k=1}^{3} \|\partial_t^k \check{q}(t)\|_{4-k}^2 + \lambda^2 \|\partial_t^4 \check{q}(t)\|_0^2 \right), \tag{4.1}$$

and $\sup_{t\in[0,T]} |\psi(t)|_\infty \le 10$.

Remark. We require the initial data, expressed in terms of $\check{q}_0$, satisfying

$$(D_t^\varphi)^i \check{q}|_{\{t=0\}} = \sigma (D_t^\varphi)^i \mathcal{H}|_{\{t=0\}} + g (D_t^\varphi)^i \psi|_{\{t=0\}}, \quad i = 0, 1, 2, 3, \quad \text{on } \Sigma.$$

We refer to [32, Section 7 and Appendix B] for the construction of such data.

Remark. Note that λ^2-weight is assigned on $\|\partial_t^4 \check{q}(t)\|_0^2$ which allows us to show that (4.1) is uniform in λ. This weight comes naturally from the second equation of (3.6) in the energy estimate.

The next theorem shows that we can pass the solution of (3.6) to that of the incompressible Euler equations without surface tension. Denote by w, p_{in} the incompressible velocity and pressure, respectively, and $x_3 = \xi(t, x')$ the moving surface boundary, the incompressible Euler equations read

$$\begin{cases} D_t^\varphi w + \nabla^\varphi \mathfrak{p} = 0, & \text{in } [0,T] \times \Omega, \\ \nabla^\varphi \cdot w = 0, & \text{in } [0,T] \times \Omega, \\ \mathfrak{p} := p_{in} + g\varphi, & \text{in } [0,T] \times \Omega, \\ \mathfrak{p} = g\xi, & \text{in } [0,T] \times \Sigma, \\ \partial_t \xi = w \cdot \mathcal{N}, & \text{in } [0,T] \times \Sigma, \\ w^3 = 0, & \text{in } [0,T] \times \Sigma_b, \\ (\xi, w)|_{t=0} = (\xi_0, w_0), \end{cases} \tag{4.2}$$

where $\mathcal{N} = (-\partial_1 \xi, -\partial_2 \xi, 1)^T$, and we still use φ to denote the extension of ξ, i.e., $\varphi(t,x) = x_3 + \chi(x_3)\xi(t,x')$ by a slight abuse of notations.

Theorem 4.3 (Incompressible and zero surface tension double limits)**.** Let $(\psi_0^{\lambda,\sigma}, v_0^{\lambda,\sigma}, \rho_0^{\lambda,\sigma} - 1)$ be the initial data[7] of (3.6) indexed by λ, σ, satisfying the compatibility conditions and

1. $|\psi_0^{\lambda,\sigma}|_\infty \le 1$, and $(\psi_0^{\lambda,\sigma}, v_0^{\lambda,\sigma}, \rho_0^{\lambda,\sigma} - 1) \to (\xi_0, w_0, 0)$ in $C^2(\Sigma) \times C^2(\Omega) \times C^1(\Omega)$ as $\lambda, \sigma \to 0$.
2. Both q and q_{in} satisfying the Rayleigh-Taylor sign condition

$$-\partial_3 q, \quad -\partial_3 q_{in} \ge c_0 > 0, \quad \text{on } \Sigma. \tag{4.3}$$

for some $c_0 > 0$.

Then it holds that

$$(\psi^{\lambda,\sigma}, v^{\lambda,\sigma}, \rho^{\lambda,\sigma} - 1) \to (\xi, w, 0), \quad \text{in } C^0([0,T], C^2(\Sigma) \times C^2(\Omega) \times C^1(\Omega)),$$

possibly after passing to a subsequence.

---

[7] Our initial data is known to be “prepared” as

$$v_0^{\lambda,\sigma} = w_0 + O(\lambda^2),$$

for each fixed σ. In fact, in [32, Appendix B] we construct $v_0^{\lambda,\sigma}$ on top of w_0. On the other hand, the incompressible limit of free-boundary Euler equations with the “general” or “ill-prepared” data remains open.

In fact, the Rayleigh-Taylor sign condition (4.3) needs only to be assumed initially, i.e.,

$$-\partial_3 q_0 \geq c_0 > 0, \quad \text{on}\,\Sigma.$$

Then through the energy (4.1), we can propagate this to a later time:

$$-\partial_3 q \geq \frac{1}{2} c_0 > 0, \quad \text{on}\,\Sigma.$$

Remark. This theorem follows directly from a uniform-in-λ, σ energy estimate of (3.6) thanks to Ascoli's theorem. In fact, the estimate for (4.1) is, apart from uniform-in-λ, also uniform-in-σ provided the Rayleigh-Taylor sign condition. We refer to Section 6 for the details.

4.1 Comparison between the different formulations of surface tension

A key observation in the proof of Theorem 4.2 is the difficulty on the surface tension mentioned in Subsection 2.3 no longer exists. As it can be seen from (4.1), the energy contributed by the surface tension reads

$$\sigma|\overline{\nabla}\partial_t^k \psi|_{4-k}^2, \quad 0 \leq k \leq 4,$$

compared to[8]

$$\sigma|\partial_t^m \eta \cdot n|_{5-m}^2, \quad 1 \leq m \leq 4.$$

In particular, top-order error terms are generated (e.g., (2.7)) while formulating the latter when derivatives land on n, which are very difficult to treat in the approximate equations. On the other hand, this issue no longer exists when estimating (4.1) simply because the mean curvature

$$\mathcal{H} = -\overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1 + |\overline{\nabla}\psi|^2}} \right)$$

does not yield a "$\cdot n$-type structure", and so we shall not have error terms parallel to (2.7). Let $\mathfrak{D} = \partial_t$ or $\overline{\partial}$. Taking $\mathfrak{D}^4$ to the first equation of (3.6) and then testing with $\mathfrak{D}^4 v$, we obtain, after integrating ∇^φ by parts, that

$$\sigma \int_\Sigma \mathfrak{D}^4 \overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1 + |\overline{\nabla}\psi|^2}} \right) (\mathfrak{D}^4 v) \cdot N \, \mathrm{d}x'$$

appears on the RHS, which is

$$\sigma \int_\Sigma \mathfrak{D}^4 \overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1 + |\overline{\nabla}\psi|^2}} \right) (\mathfrak{D}^4 \partial_t \psi) \, \mathrm{d}x' - \sigma \int_\Sigma \mathfrak{D}^4 \overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1 + |\overline{\nabla}\psi|^2}} \right) (\overline{v} \cdot \overline{\nabla}) \mathfrak{D}^4 \psi \, \mathrm{d}x' := J_1 + J_2,$$

up to lower-order terms. Integrating $\overline{\nabla}$ by parts in J_2, we have

$$J_2 \overset{L}{=} \sigma \int_\Sigma \left(\frac{\mathfrak{D}^4 \overline{\nabla}\psi}{\sqrt{1 + |\overline{\nabla}\psi|^2}} \right) \cdot (\overline{v} \cdot \overline{\nabla}) \overline{\nabla} \mathfrak{D}^4 \psi \, \mathrm{d}x'$$

Then, it can be seen that the top order terms are canceled by symmetry after integrating $\overline{\partial} \cdot \overline{\nabla}$ by parts.

In addition, integrating $\overline{\nabla}\cdot$ by parts in J_1, we get

$$-\sigma \int_\Sigma \left(\frac{\mathfrak{D}^4 \overline{\nabla}\psi}{\sqrt{1 + |\overline{\nabla}\psi|^2}} - \frac{\overline{\nabla}\psi \cdot \mathfrak{D}^4 \overline{\nabla}\psi}{\left(\sqrt{1 + |\overline{\nabla}\psi|^2}\right)^3} \overline{\nabla}\psi \right) \cdot \overline{\nabla} \mathfrak{D}^4 \partial_t \psi \, \mathrm{d}x'$$

[8]The energy norm induced by the surface tension ties to the second fundamental form of Σ_t. This corresponds to the *normal component* of the top order term of η in the Lagrangian coordinates. Nevertheless, in the coordinates induced by ψ, this is just the top order term of $\overline{\nabla}\psi$.

at the leading order, which contributes to, after moving to the LHS, that

$$\frac{\sigma}{2}\frac{d}{dt}\int_\Sigma\left(\frac{|\mathfrak{D}^4\overline{\nabla}\psi|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}}-\frac{|\overline{\nabla}\psi\cdot\mathfrak{D}^4\overline{\nabla}\psi|^2}{\left(\sqrt{1+|\overline{\nabla}\psi|^2}\right)^3}\right)\mathrm{d}x', \quad \text{and} \quad \int_\Sigma\left(\frac{|\mathfrak{D}^4\overline{\nabla}\psi|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}}-\frac{|\overline{\nabla}\psi\cdot\mathfrak{D}^4\overline{\nabla}\psi|^2}{\left(\sqrt{1+|\overline{\nabla}\psi|^2}\right)^3}\right)\mathrm{d}x' \geq \int_\Sigma\frac{|\mathfrak{D}^4\overline{\nabla}\psi|^2}{\left(\sqrt{1+|\overline{\nabla}\psi|^2}\right)^3}\mathrm{d}x', \tag{4.4}$$

where

$$\int_\Sigma\frac{|\mathfrak{D}^4\overline{\nabla}\psi|^2}{\left(\sqrt{1+|\overline{\nabla}\psi|^2}\right)^3}\mathrm{d}x' \geq \int_\Sigma|\mathfrak{D}^4\overline{\nabla}\psi|^2\,\mathrm{d}x'.$$

In (4.4) above, we applied

$$\frac{|\mathbf{a}|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}}-\frac{|\overline{\nabla}\psi\cdot\mathbf{a}|^2}{\left(\sqrt{1+|\overline{\nabla}\psi|^2}\right)^3}\geq\frac{|\mathbf{a}|^2}{\left(\sqrt{1+|\overline{\nabla}\psi|^2}\right)^3}. \tag{4.5}$$

Note that all error terms mentioned above are of lower order and thus we have no problem controlling them.

5 Major ideas of the proof I: The approximate κ-equations

Related to the Lagrangian formulation, we require to set up a sequence of approximate equations, indexed by $\kappa>0$, which are asymptotically consistent with (3.6). This is necessary when proving the existence of the free-boundary problem since the a priori estimate no longer holds for the linearized equations due to loss of symmetry. This can be seen by re-doing (4.4) in the linearized setting. In particular, the linearized boundary condition of q reads

$$q=-\sigma\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\mathring{\psi}|^2}}\right), \quad \text{on } \Sigma,$$

where $\mathring{\psi}$ is a given function. Invoking the linearized boundary condition $\partial_t\psi=v\cdot\mathring{N}$, the LHS of (4.4) becomes

$$\frac{\sigma}{2}\frac{d}{dt}\int_\Sigma\left(\frac{|\mathfrak{D}^4\overline{\nabla}\psi|^2}{\sqrt{1+|\overline{\nabla}\mathring{\psi}|^2}}-\frac{(\overline{\nabla}\psi\cdot\mathfrak{D}^4\overline{\nabla}\psi)(\overline{\nabla}\mathring{\psi}\cdot\mathfrak{D}^4\overline{\nabla}\mathring{\psi})}{\left(\sqrt{1+|\overline{\nabla}\mathring{\psi}|^2}\right)^3}\right)\mathrm{d}x',$$

and the inequality (4.5) is not applicable here. Also, the second term in the above integral yield a loss of (tangential) derivatives.

To solve this issue, we design approximate equations involving *tangentially smoothed* nonlinear coefficients, through which we gain regularity in the tangential direction.

Definition 5.1 (Tangential smoothing)**.** Let $\zeta=\zeta(x')\in C_c^\infty(\mathbb{R}^2)$ be the standard bump function supported in the closed unit ball. We define $\zeta_\kappa(x')=\frac{1}{\kappa^2}\zeta(\frac{x'}{\kappa})$, for each $\kappa>0$. Let $f:\mathbb{R}^2\to\mathbb{R}$. We set

$$\Lambda_\kappa f(x')=\int_{\mathbb{R}^2}\zeta_\kappa(x'-z')f(z')\,dz'.$$

Also, for each $g:\mathbb{R}^3\to\mathbb{R}$, we set

$$\Lambda_\kappa g(x',z)=\int_{\mathbb{R}^2}\zeta_\kappa(x-z')g(z',x_3)\,dz'.$$

Notation 5.1 (Smoothed derivatives)**.** Let $\widetilde{\psi}:=\Lambda_\kappa^2\psi$, $\widetilde{\varphi}:=x_3+\chi(x_3)\widetilde{\psi}(t,x')$, and $\widetilde{\mathbf{N}}=(-\partial_1\widetilde{\varphi},-\partial_2\widetilde{\varphi},1)^T$. We define:

i. (Smoothed Eulerian time derivative) $\partial_t^{\widetilde{\varphi}}:=\partial_t-(\partial_t\varphi)(\partial_3\widetilde{\varphi})^{-1}\partial_3$.
ii. (Smoothed Eulerian spatial derivatives) $\nabla_\tau^{\widetilde{\varphi}}=\partial_\tau^{\widetilde{\varphi}}:=\partial_\tau-(\partial_\tau\widetilde{\varphi})(\partial_3\widetilde{\varphi})^{-1}\partial_3$, $\tau=1,2$, and $\nabla_3^{\widetilde{\varphi}}=\partial_3^{\widetilde{\varphi}}:=(\partial_3\widetilde{\varphi})^{-1}\partial_3$.

iii. (Smoothed material derivative) $D_t^{\widetilde{\varphi}} := \partial_t + \overline{v}\cdot\overline{\partial} + (\partial_3\widetilde{\varphi})^{-1}(v\cdot\widetilde{\mathbf{N}} - \partial_t\varphi)\partial_3$.

Then, with $\widetilde{N} = (-\partial_1\widetilde{\psi}, -\partial_2\widetilde{\psi}, 1)^T$, we set the approximate κ-equations as

$$\begin{cases} \rho D_t^{\widetilde{\varphi}} v + \nabla^{\widetilde{\varphi}}\check{q} = -(\rho-1)ge_3, & \text{in } [0,T]\times\Omega,\\ \lambda^2 D_t^{\widetilde{\varphi}}\check{q} + \nabla^{\widetilde{\varphi}}\cdot v = g\lambda^2 v^3, & \text{in } [0,T]\times\Omega,\\ \partial_t\psi = v\cdot\widetilde{N}, & \text{on } [0,T]\times\Sigma,\\ \check{q} = g\widetilde{\psi} - \sigma\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\widetilde{\psi}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}}\right) + \kappa^2(1-\overline{\Delta})\partial_t\psi, & \text{on } [0,T]\times\Sigma,\\ v^3 = 0, & \text{on } [0,T]\times\Sigma_b,\\ (\psi, v, \check{q})|_{t=0} = (\psi_{\kappa,0}, v_{\kappa,0}, \check{q}_{\kappa,0}). \end{cases} \tag{5.1}$$

Remark. Note that there is an artificial viscosity $\kappa^2(1-\overline{\Delta})\partial_t\psi$ that appears on the boundary condition of $\check{q}$. This is necessary since mismatched terms involving $\partial_t\overline{\partial}^4(\widetilde{\varphi}-\varphi)$ shall appear in the energy estimate of (5.1). In particular, the following estimate holds:

$$\|\partial_t\overline{\partial}^4(\widetilde{\varphi}-\varphi)\|_0 \lesssim |\partial_t\overline{\partial}^4(\widetilde{\psi}-\psi)|_0 \lesssim \kappa|\overline{\nabla}\overline{\partial}^4\partial_t\psi|_0, \tag{5.2}$$

where the last inequality is deduced from the properties of tangential smoothing. After integrating in time, this can be controlled directly from the κ-weighted energy contributed by $\kappa^2(1-\overline{\Delta})\partial_t\psi$.

Remark. Compared to the remark after Theorem 4.2, the compatibility conditions have changed due to the presence of artificial viscosity. As a consequence, for each fixed $\kappa > 0$, we have to construct data $(\psi_{\kappa,0}, v_{\kappa,0}, \check{q}_{\kappa,0}) \in H^5(\Sigma)\times H^4(\Omega)\times H^4(\Omega)$ that converges to $(\psi_0, v_0, \check{q}_0)$ as $\kappa\to 0$. The idea of constructing $(\psi_{\kappa,0}, v_{\kappa,0}, \check{q}_{\kappa,0})$ is identical to the construction of the prepared data $(\psi_0, v_0, \check{q}_0)$ from the incompressible data.

The setup of the approximate system 5.1 is more dedicated compared to the approximate system in Lagrangian coordinates. This is due that that $D_t = \partial_t$ in Lagrangian coordinates and thus the kinematic boundary condition does not show up explicitly. On the other hand, to set (5.1) up properly, we have to make sure the smoothed kinematic boundary condition is designed so that: i). The material derivative $D_t^{\widetilde{\varphi}}|_\Sigma \in T(\Sigma)$, which is indeed the case since $(v\cdot\widetilde{\mathbf{N}} - \partial_t\varphi)|_\Sigma = v\cdot\widetilde{N} - \partial_t\psi = 0$. ii). The following Raynold-type transport theorem yields no top-order terms on the boundary.

Lemma 5.2. Let f be a smooth function defined on $[0,T]\times\Omega$. Then

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|f|^2\partial_3\widetilde{\varphi}\,\mathrm{d}x = \int_\Omega \rho(D_t^{\widetilde{\varphi}} f)f\partial_3\widetilde{\varphi}\,\mathrm{d}x + \frac{1}{2}\int_\Omega \rho|f|^2\partial_3\partial_t(\widetilde{\varphi}-\varphi)\,\mathrm{d}x.$$

This is one of the key lemmas for proving the energy estimate of the κ-equations, whose proof can be found in Appendix A of [32]. Specifically, the term

$$\frac{1}{2}\int_\Sigma \rho|f|^2(v\cdot\widetilde{N} - \partial_t\psi)\,\mathrm{d}x' \tag{5.3}$$

shows up in the proof, which is 0 because of the kinematic boundary condition. We shall assign $f = \mathfrak{D}^4 v$ when applying Lemma 5.2, and thus (5.3) would have no control unless $\partial_t\psi = v\cdot\widetilde{N}$.

6 Major ideas of the proof II: The uniform energy estimate of the approximate κ-equations

In this section, we survey the energy estimate of the approximate κ-problem (5.1). In particular, we prove a uniform energy estimate in κ. This then allows us to pass the κ-solution to the solution of the original equations (3.6). Meanwhile, we show that the κ-energy estimate is uniform in λ, and also uniform in σ if the Rayleigh-Taylor sign condition (4.3) is further assumed.

Theorem 6.1. Let

$$\begin{aligned} E^\kappa(t) = \sum_{k=0}^4 \left(\|\partial_t^k v(t)\|_{4-k}^2 + \sigma|\overline{\nabla}\partial_t^k\Lambda_\kappa\psi(t)|_{4-k}^2\right) + \left(\|\partial\check{q}(t)\|_3^2 + \sum_{k=1}^3\|\partial_t^k\check{q}(t)\|_{4-k}^2 + \lambda^2\|\partial_t^4\check{q}(t)\|_0^2\right)\\ + \sum_{k=0}^4\int_0^t |\kappa\partial_t^{k+1}\psi(s)|_{5-k}^2\,ds. \end{aligned} \tag{6.1}$$

Then, for each $\sigma > 0$ fixed, there exists a $T_1 > 0$, independent of κ and λ, such that

$$E^{\kappa}(t) \leq C(\frac{1}{\sigma}, E^{\kappa}(0)), \quad t \in [0, T_1]. \tag{6.2}$$

Moreover, let

$$\begin{aligned} E^{\kappa,\sigma}(t) = \sum_{k=0}^{4} \left(\|\partial_t^k v(t)\|_{4-k}^2 + \sigma |\overline{\nabla} \partial_t^k \Lambda_\kappa \psi(t)|_{4-k}^2 + |\partial_t^k \Lambda_\kappa \psi(t)|_{4-k}^2 \right) + \left(\|\partial \check{q}(t)\|_3^2 + \sum_{k=1}^{3} \|\partial_t^k \check{q}(t)\|_{4-k}^2 + \lambda^2 \|\partial_t^4 \check{q}(t)\|_0^2 \right) \\ + \sum_{k=0}^{4} \int_0^t |\kappa \partial_t^{k+1} \psi(s)|_{5-k}^2 \, ds. \end{aligned} \tag{6.3}$$

Assume the Rayleigh-Taylor sign condition (4.3) holds, then there exists a $T_2 > 0$, independent of κ, λ and σ, such that

$$E^{\kappa,\sigma}(t) \leq C(E^{\kappa,\sigma}(0)), \quad t \in [0, T_2]. \tag{6.4}$$

We shall take $T = \min(T_1, T_2)$ in the sequel. Also, we have, in fact, $E^{\kappa}(0)$ is comparable with $E^{\kappa,\sigma}(0)$ and they both depend on the data of (5.1). Because of the Grönwall's theorem, the proof of (6.2) is a direct consequence of

$$E^{\kappa}(T) \leq \mathcal{P}_0^{\kappa} + \int_0^T P(E^{\kappa}(t))\, \mathrm{d}t, \tag{6.5}$$

where $P(\cdots)$ stands for a generic polynomial in its arguments, and $\mathcal{P}_0^{\kappa} := P(E^{\kappa}(0))$. Similarly, (6.4) follows from

$$E^{\kappa,\sigma}(T) \leq \mathcal{P}_0^{\kappa,\sigma} + \int_0^T P(E^{\kappa,\sigma}(t))\, \mathrm{d}t. \tag{6.6}$$

We will sketch the proof of Theorem 6.1 in the remaining of this section by establishing (6.5) and (6.6).

6.1 The Hodge-type div-curl estimate

For $j = 1, 2, 3$, since we can still express $\partial_j^{\varphi} = \mathcal{A}^{ij}\partial_i$, where $\mathcal{A}$ is defined in (3.1), we can analogously denote $\partial_j^{\widetilde{\varphi}}$ in a similar fashion. This indicates that (2.5) holds true under the current coordinates system, and becomes

$$\|X\|_{H^s(\Omega)}^2 \leq C(|\overline{\partial}\widetilde{\psi}|_{W^{1,\infty}(\Omega)}, |\widetilde{\psi}|_s^2) \left(\|X\|_0^2 + \|\nabla^{\widetilde{\varphi}} \cdot X\|_{s-1}^2 + \|\nabla^{\widetilde{\varphi}} \times X\|_{s-1}^2 + \|\overline{\partial}^s X\|_0^2 \right). \tag{6.7}$$

Setting $X = \partial_t^k v$ and $s = 4 - k$ for $k = 0, 1, 2, 3$, (6.7) implies that the control of $\|\partial_t^k v\|_{4-k}^2$ reduces to that of

$$\|\nabla^{\widetilde{\varphi}} \cdot \partial_t^k v\|_{3-k}^2, \quad \|\nabla^{\widetilde{\varphi}} \times \partial_t^k v\|_{3-k}^2, \quad |\overline{\partial}^{4-k} \partial_t^k v|_0^2.$$

The third term contributes to the tangential energy estimate that will be discussed in Subsection 6.3. The first and second terms are simple to control. We control the second term by studying the evolution equation of $\nabla^{\widetilde{\varphi}} \times v$ (and its time-differentiated version) by taking $\nabla^{\widetilde{\varphi}}\times$ to the momentum equation, i.e., the first equation of (5.1). Moreover, we use the continuity equation $\nabla^{\widetilde{\varphi}} \cdot v = -\lambda^2 D_t^{\widetilde{\varphi}} \check{q} + g\lambda^2 v^3$ to reduce $\|\nabla^{\widetilde{\varphi}} \cdot \partial_t^k v\|_{3-k}$ to

$$\|\lambda^2 \partial_t^k D_t^{\widetilde{\varphi}} \check{q}\|_{3-k} \leq \lambda^2 \left(\|\partial_t^{k+1} \check{q}\|_{3-k} + \|\partial_t^k (\overline{v} \cdot \overline{\partial}) \check{q}\|_{3-k} + \|\partial_t^k ((v \cdot \widetilde{\mathbf{N}} - \partial_t \varphi) \partial_3^{\widetilde{\varphi}} \check{q})\|_{3-k} \right).$$

Since

$$\|\partial_t^k ((v \cdot \widetilde{\mathbf{N}} - \partial_t \varphi) \partial_3^{\widetilde{\varphi}} \check{q})\|_{3-k} \stackrel{L}{=} \|(v \cdot \widetilde{\mathbf{N}} - \partial_t \varphi) \partial_3^{\widetilde{\varphi}} \partial_t^k \check{q}\|_{3-k}$$

and $v \cdot \widetilde{\mathbf{N}} - \partial_t \varphi = 0$ on Σ and Σ_t, we may set $\omega = \omega(x_3)$ to be a smooth function on $[-b, 0]$, satisfying $\omega(-b) = \omega(0) = 0$ and thus comparable to $v \cdot \widetilde{\mathbf{N}} - \partial_t \varphi$. Then, we may have the expanded tangential derivatives, denoted by $\mathcal{T}$, consisting of

$$\partial_t, \quad \overline{\partial}, \quad \text{and } \omega(x_3)\partial_3. \tag{6.8}$$

Thus, it can be seen that, after expanding $D_t^{\widetilde{\varphi}}$, $\lambda^2 \|\partial_t^k D_t^{\widetilde{\varphi}} \check{q}\|_{3-k}$ contributes to terms with $k+1$ tangential derivatives. This, combined with the procedure introduced in the next Subsection, allow us to reduce $\|\nabla^{\widetilde{\varphi}} \cdot \partial_t^k v\|_{3-k}$ to the tangential energy that we are going to bound in Subsection 6.3.

6.2 Reduction of $\check{q}$

The momentum equation yields

$$\partial_3^{\widetilde{\varphi}}\check{q} = -\rho\left(\partial_t v_3 + \overline{v}\cdot\overline{\partial}v_3 + (v\cdot\widetilde{\mathbf{N}} - \partial_t\varphi)\partial_3^{\widetilde{\varphi}}v_3\right) - g(\rho-1),$$

which implies

$$\partial_3\check{q} = -\rho(\partial_3\widetilde{\varphi})\left(\partial_t v_3 + \overline{v}\cdot\overline{\partial}v_3 + (v\cdot\widetilde{\mathbf{N}} - \partial_t\varphi)\partial_3^{\widetilde{\varphi}}v_3\right) - g(\partial_3\widetilde{\varphi})(\rho-1). \tag{6.9}$$

On the other hand, for $\tau = 1,2$, the momentum equation gives

$$\partial_\tau^{\widetilde{\varphi}}\check{q} = -\rho\left(\partial_t v_\tau + \overline{v}\cdot\overline{\partial}v_\tau + (v\cdot\widetilde{\mathbf{N}} - \partial_t\varphi)\partial_3^{\widetilde{\varphi}}v_\tau\right),$$

and thus

$$\begin{aligned}\partial_\tau\check{q} &= (\partial_\tau\widetilde{\varphi})\partial_3^{\widetilde{\varphi}}\check{q} - \rho\left(\partial_t v_\tau + \overline{v}\cdot\overline{\partial}v_\tau + (v\cdot\widetilde{\mathbf{N}} - \partial_t\varphi)\partial_3^{\widetilde{\varphi}}v_\tau\right)\\ &= -\rho(\partial_\tau\widetilde{\varphi})\left(\partial_t v_3 + \overline{v}\cdot\overline{\partial}v_3 + (v\cdot\widetilde{\mathbf{N}} - \partial_t\varphi)\partial_3^{\widetilde{\varphi}}v_3\right) - \rho\left(\partial_t v_\tau + \overline{v}\cdot\overline{\partial}v_\tau + (v\cdot\widetilde{\mathbf{N}} - \partial_t\varphi)\partial_3^{\widetilde{\varphi}}v_\tau\right) - g(\partial_\tau\widetilde{\varphi})(\rho-1).\end{aligned} \tag{6.10}$$

Recall the definition of the extended tangential derivative $\mathcal{T}$ in (6.8), from (6.9) and (6.10) we can see that $\partial_j\check{q}$ is reduced to $D_t^{\widetilde{\varphi}}v_j$ consisting of $\mathcal{T}v_j$, for $j = 1,2,3$, at the top order. This can be generalized to higher-order derivatives of $\check{q}$: Denote by $D = \partial_t$ or ∂, then for $k \geq 1$, $D^k\partial\check{q}$ is reduced to $D^kD_t^{\widetilde{\varphi}}v$, which consists of $D^k\mathcal{T}v$ at the leading order. In short, by combining the techniques introduced in Subsections 6.3 and 6.2, we can reduce the full norms $\|v\|_4$ and $\|\check{q}\|_4$ to the tangential energy.

On the other hand, the analysis in Subsections 6.1–6.2 determines the appropriate λ^2-weight that we need to assign in the energies $E^\kappa(t)$ and $E^{\kappa,\sigma}(t)$ defined respectively as (6.1) and (6.3). Since the reduction scheme is derived from the momentum equation which has no λ^2-weight, the L^2-norms of $D^4\check{q}$ should not be weighted as long as D^4 consists of at least one spatial derivative. Nevertheless, the reduction scheme fails if no spatial derivative is attached to $\check{q}$. We therefore must assign λ^2-weight to $\|\partial_t^4\check{q}\|_0^2$ in both $E^\kappa(t)$ and $E^{\kappa,\sigma}(t)$.

6.3 The tangential energy estimate

Let $\alpha = (\alpha_0,\alpha_1,\alpha_2,\alpha_3)$ be a multi-index with $|\alpha| \leq 4$. We denote by $\mathcal{T}^\alpha = \mathcal{T}_0^{\alpha_0}\mathcal{T}_1^{\alpha_1}\mathcal{T}_2^{\alpha_2}\mathcal{T}_3^{\alpha_3}$ to be the mixed general tangential derivatives with order $|\alpha|$, where

$$\mathcal{T}_0 = \partial_t,\quad \mathcal{T}_1 = \partial_1,\quad \mathcal{T}_2 = \partial_2,\quad \mathcal{T}_3 = \omega(x_3)\partial_3,$$

and we recall that $\mathcal{T}_3$ is constructed in Subsection 6.1 and it is set to be comparable to with $(v\cdot\widetilde{\mathbf{N}} - \partial_t\varphi)\partial_3$. Also, notice that since $\mathcal{T}_3$ vanishes on Σ, we have $\mathcal{T}|_\Sigma = \{\partial_t,\overline{\partial}\}$. This applies also to Σ_b.

Notation 6.2 (Commutators)**.** Let f,g be smooth functions on $[0,T]\times\Omega$.

i. (Standard commutator) $[\mathcal{T}^\alpha, f]g := \mathcal{T}^\alpha(fg) - f\mathcal{T}^\alpha g$.
ii. (Symmetric commutator) $[\mathcal{T}^\alpha, f, g] := \mathcal{T}^\alpha(fg) - (\mathcal{T}^\alpha f)g - f(\mathcal{T}^\alpha g)$.

Theorem 6.3. Let $\mathcal{T}^\alpha$ be defined as above with $|\alpha| = 4$. For each fixed $\sigma > 0$, there exists $T > 0$, independent of κ and λ, such that

$$\|\mathcal{T}^\alpha v(T)\|_0^2 + \lambda^2\|\mathcal{T}^\alpha\check{q}(T)\|_0^2 + \sigma|\overline{\nabla}\mathcal{T}^\alpha\Lambda_\kappa\psi(T)|_0^2 + \int_0^T|\kappa\mathcal{T}^\alpha\partial_t\psi(t)|_1^2\,\mathrm{d}t \lesssim_{\sigma^{-1}} \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t. \tag{6.11}$$

Furthermore, if $-\partial_3 q \geq c_0 > 0$ holds on Σ, we have

$$\|\mathcal{T}^\alpha v(T)\|_0^2 + \lambda^2\|\mathcal{T}^\alpha\check{q}(T)\|_0^2 + \sigma|\overline{\nabla}\mathcal{T}^\alpha\Lambda_\kappa\psi(T)|_0^2 + |\mathcal{T}^\alpha\Lambda_\kappa\psi(T)|_0^2 + \int_0^T|\kappa\mathcal{T}^\alpha\partial_t\psi(t)|_1^2\,\mathrm{d}t \leq \mathcal{P}_0^{\kappa,\sigma} + \int_0^T P(E^{\kappa,\sigma}(t))\,\mathrm{d}t, \tag{6.12}$$

where $T > 0$ is also independent of σ.

Remark. The tangential estimate (6.11) is associated to (6.5) after the div-curl analysis and reduction of $\check{q}$, whereas (6.12) is associated to (6.6) analogously.

The first step of the proof of Theorem 6.3 is to test the $\mathcal{T}^\alpha$-differentiated (5.1) with $\mathcal{T}^\alpha v$. However, when commuting $\mathcal{T}^\alpha$ through $\nabla^{\widetilde{\varphi}}\check{q}$, the commutator

$$[\mathcal{T}^\alpha, \nabla^{\widetilde{\varphi}}]\check{p} = [\mathcal{T}^\alpha, \mathcal{A}]\partial\check{q}$$

which yields $-\mathcal{T}^\alpha\left(\frac{\partial_j\widetilde{\varphi}}{\partial_3\widetilde{\varphi}}\right)\partial\check{q} \overset{L}{=} -\frac{\mathcal{T}^\alpha\partial_j\widetilde{\varphi}}{\partial_3\widetilde{\varphi}}\partial\check{q}$ when $j=1,2$. Note that this requires $\sqrt{\sigma}|\overline{\nabla}\partial_t^k\Lambda_\kappa\psi|_{4-k}$ to control if $|\alpha|=4, \alpha_3=0$, and thus it cannot be done uniformly in σ! To overcome this issue, we invoke Alinhac's good unknowns.

Definition 6.1 (Alinhac's good unknown)**.** Let f be a smooth function on $[0,T]\times\Omega$. We define

$$\mathbf{F} := \mathcal{T}^\alpha f - \partial_3^{\widetilde{\varphi}} f \mathcal{T}^\alpha\widetilde{\varphi}. \tag{6.13}$$

to be the Alinhac's good unknown (associated with $\mathcal{T}^\alpha$) of f.

A direct computation shows that Alinhac's good unknown enjoys the following commutator properties:

i. (Commuting with $\nabla^{\widetilde{\varphi}}$)

$$\mathcal{T}^\alpha\partial_i^{\widetilde{\varphi}} f = \partial_i^{\widetilde{\varphi}}\mathbf{F} + C_i(f), \tag{6.14}$$

where $C_i(f) = \partial_3^{\widetilde{\varphi}}\partial_i^{\widetilde{\varphi}} f\mathcal{T}^\alpha\widetilde{\varphi} + C_i'(f)$, and

$$C_i'(f) = -\left[\mathcal{T}^\alpha, \frac{\partial_i\widetilde{\varphi}}{\partial_3\widetilde{\varphi}}, \partial_3 f\right] - \partial_3 f\left[\mathcal{T}^\alpha, \partial_i\widetilde{\varphi}, \frac{1}{\partial_3\widetilde{\varphi}}\right] + \partial_i\widetilde{\varphi}\partial_3 f\left[\mathcal{T}^{\alpha-\gamma}, \frac{1}{(\partial_3\widetilde{\varphi})^2}\right]\mathcal{T}^\gamma\partial_3\widetilde{\varphi} - \frac{\partial_i\widetilde{\varphi}}{\partial_3\widetilde{\varphi}}[\mathcal{T}^\alpha, \partial_3]f + \frac{\partial_i\widetilde{\varphi}}{(\partial_3\widetilde{\varphi})^2}\partial_3 f[\mathcal{T}^\alpha, \partial_3]\widetilde{\varphi},$$

$i=1,2$, with $|\gamma|=1$, and

$$C_3'(f) = \left[\mathcal{T}^\alpha, \frac{1}{\partial_3\widetilde{\varphi}}, \partial_3 f\right] + \partial_3 f\left[\mathcal{T}^{\alpha-\gamma}, \frac{1}{(\partial_3\widetilde{\varphi})^2}\right]\mathcal{T}^\gamma\partial_3\widetilde{\varphi} - \frac{1}{\partial_3\widetilde{\varphi}}[\mathcal{T}^\alpha, \partial_3]f + \frac{1}{(\partial_3\widetilde{\varphi})^2}\partial_3 f[\mathcal{T}^\alpha, \partial_3]\widetilde{\varphi}.$$

ii. (Commuting with $D_t^{\widetilde{\varphi}}$)

$$\mathcal{T}^\alpha D_t^{\widetilde{\varphi}} f = D_t^{\widetilde{\varphi}}\mathbf{F} + D(f) + \mathcal{E}(f), \tag{6.15}$$

where $D(f) = D_t^{\widetilde{\varphi}}\partial_3^{\widetilde{\varphi}} f\mathcal{T}^\alpha\widetilde{\varphi} + D'(f)$, and

$$\begin{aligned} D'(f) = [\mathcal{T}^\alpha, \overline{v}]\cdot\overline{\partial} f + \left[\mathcal{T}^\alpha, \frac{1}{\partial_3\widetilde{\varphi}}(v\cdot\widetilde{\mathbf{N}} - \partial_t\varphi), \partial_3 f\right] + \left[\mathcal{T}^\alpha, v\cdot\widetilde{\mathbf{N}} - \partial_t\varphi, \frac{1}{\partial_3\widetilde{\varphi}}\right]\partial_3 f + \frac{1}{\partial_3\widetilde{\varphi}}[\mathcal{T}^\alpha, v]\cdot\widetilde{\mathbf{N}}\partial_3 f \\ -(v\cdot\widetilde{\mathbf{N}} - \partial_t\varphi)\partial_3 f\left[\overline{\partial}^{\alpha-\gamma}, \frac{1}{(\partial_3\widetilde{\varphi})^2}\right]\mathcal{T}^\gamma\partial_3\widetilde{\varphi} + \frac{1}{\partial_3\widetilde{\varphi}}(v\cdot\widetilde{\mathbf{N}} - \partial_t\varphi)[\mathcal{T}^\alpha, \partial_3]f + (v\cdot\widetilde{\mathbf{N}} - \partial_t\varphi)\frac{\partial_3 f}{(\partial_3\widetilde{\varphi})^2}[\mathcal{T}^\alpha, \partial_3]\widetilde{\varphi}, \end{aligned}$$

$$\mathcal{E}(f) = \partial_t\mathcal{T}^\alpha(\widetilde{\varphi} - \varphi)\partial_3^{\widetilde{\varphi}} f.$$

Remark. Despite being long, it is not hard to see that neither $C_i(f)$ nor $D(f)$ have top-order terms in $\widetilde{\varphi}$. In other words, using Alinhac good unknowns allows us to avoid unnecessarily applying the surface tension energy without $\sqrt{\sigma}$. In addition, $\mathcal{E}(f)$ records the error term generated by the mismatch, which has been treated in the remark under (5.1).

Denote by $\mathbf{V}$ and $\mathbf{Q}$ the Alinhac good unknowns of v and $\check{q}$, respectively, we then infer from (6.14) and (6.15) that

$$\begin{aligned} \rho D_t^{\widetilde{\varphi}}\mathbf{V}_i + \nabla_i^{\widetilde{\varphi}}\mathbf{Q} = R_i^1, \quad \text{in } [0,T]\times\Omega, \\ \lambda^2 D_t^{\widetilde{\varphi}}\mathbf{Q} + \nabla^{\widetilde{\varphi}}\cdot\mathbf{V} = R^2 - C_i(v^i), \quad \text{in } [0,T]\times\Omega, \end{aligned} \tag{6.16}$$

where

$$R_i^1 = -[\mathcal{T}^\alpha, \rho]D_t^{\widetilde{\varphi}} v_i - \rho(D(v_i) + \mathcal{E}(v_i)) - C_i(\check{q}),$$

and

$$R^2 = -\lambda^2(D(\check{q}) + \mathcal{E}(\check{q})) + \lambda^2 g\mathcal{T}^\alpha v_3.$$

Particularly, both R_i^1 and R^2 contribute only to harmless error terms, whose control is straightforward.

On the other hand, we need to assign boundary conditions to $\mathbf{V}$ and $\mathbf{Q}$. Note that $\mathcal{T}_3|_\Sigma = 0$, and thus $\mathcal{T}^\alpha|_\Sigma = \partial_t^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}$ with $\alpha_0+\alpha_1+\alpha_2 \le 4$, and we assume the equality throughout. This indicates that $\mathcal{T}^\alpha|_\Sigma = \mathfrak{D}^\alpha$, where $\mathfrak{D}_0 = \partial_t, \mathfrak{D}_1 = \partial_1, \mathfrak{D}_2 = \partial_2$. Since $\partial_3\widetilde{\varphi}|_\Sigma = 1$, we have

$$\mathbf{Q}|_\Sigma = (\mathfrak{D}^\alpha\check{q} - \partial_3^{\widetilde{\varphi}}\check{q}\mathfrak{D}^\alpha\widetilde{\varphi})|_\Sigma = \mathfrak{D}^\alpha(q+g\widetilde{\psi}) - (\partial_3 q + g)\mathfrak{D}^\alpha\widetilde{\psi} = \mathfrak{D}^\alpha q - \partial_3 q\mathfrak{D}^\alpha\widetilde{\psi}.$$

So,

$$\mathbf{Q} = -\sigma\mathfrak{D}^\alpha\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\widetilde{\psi}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}}\right) + \kappa^2(1-\overline{\Delta})\mathfrak{D}^\alpha\partial_t\psi - \partial_3 q\mathfrak{D}^\alpha\widetilde{\psi}, \quad \text{on } \Sigma. \tag{6.17}$$

Moreover, the $\mathfrak{D}^\alpha$-differentiated $\partial_t\psi = v\cdot(-\partial_1\widetilde{\psi}, -\partial_2\widetilde{\psi}, 1)^T$ is

$$\partial_t\mathfrak{D}^\alpha\psi = -(\overline{v}\cdot\overline{\partial})\mathfrak{D}^\alpha\widetilde{\psi} + \mathbf{V}\cdot\widetilde{N} + S_1, \quad \text{on } \Sigma, \tag{6.18}$$

where $S_1 = \partial_3 v\cdot\widetilde{N}\mathfrak{D}^\alpha\widetilde{\psi} + \sum_{|\beta_1|+|\beta_2|=4, |\beta_{1,2}|\ge 1}\mathfrak{D}^{\beta_1}v\cdot\mathfrak{D}^{\beta_2}\widetilde{N}$, and

$$\mathbf{V}^3 = 0, \quad \text{on } \Sigma_b. \tag{6.19}$$

Remark. $\mathbf{V}^3 = 0$ on the flat bottom is a direct consequence of $v^3 = 0$ and $\overline{\partial}^\alpha\widetilde{\psi} = 0$ on Σ_b. Thanks to this condition, the flat bottom plays no role in the upcoming tangential energy estimate.

6.3.1 The tangential energy identity

Testing $\mathbf{V}$ with (6.16)–(6.19), and using Lemma 5.2, we obtain the energy identity:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|\mathbf{V}|^2\partial_3\widetilde{\varphi}\,\mathrm{d}x = -\int_\Omega \mathbf{V}\cdot(\nabla^{\widetilde{\varphi}}\mathbf{Q})\,\mathrm{d}x + \frac{1}{2}\int_\Omega \rho|\mathbf{V}|^2\partial_3\partial_t(\widetilde{\varphi}-\varphi)\,\mathrm{d}x + \int_\Omega \mathbf{V}\cdot R^1\partial_3\widetilde{\varphi}\,\mathrm{d}x. \tag{6.20}$$

The final two terms are easy to control. In particular, the control of the second term relies on

$$\|\partial_3\partial_t(\widetilde{\varphi}-\varphi)\|_\infty \lesssim |\partial_t(\widetilde{\psi}-\psi)|_\infty \lesssim \sqrt{\kappa}|\overline{\partial}\partial_t\psi|_{0.5},$$

thanks to the tangential smoothing. Here, $\sqrt{\kappa}$-weight is inessential here since $|\overline{\partial}\partial_t\psi|_{0.5}$ is of very lower order.

Integrating $\nabla^{\widetilde{\varphi}}$ by parts in the first term and using (6.19),

$$-\int_\Omega \mathbf{V}\cdot(\nabla^{\widetilde{\varphi}}\mathbf{Q})\,\mathrm{d}x = \int_\Omega(\nabla^{\widetilde{\varphi}}\cdot\mathbf{V})\mathbf{Q}\partial_3\widetilde{\varphi}\,\mathrm{d}x - \int_\Sigma \mathbf{Q}(\mathbf{V}\cdot\widetilde{N})\,\mathrm{d}x'.$$

Invoking the second equation of (6.16), the first term contributes to the energy term $\frac{1}{2}\frac{d}{dt}\lambda^2\int_\Omega|\mathbf{Q}|^2\partial_3\widetilde{\varphi}\,\mathrm{d}x$, together with

$$-\int_\Omega \mathbf{Q}C_i(v^i)\partial_3\widetilde{\varphi}\,\mathrm{d}x, \tag{6.21}$$

and other easy-to-control terms. We recall that the term $-\left[\mathcal{T}^\alpha, \frac{\partial_i\widetilde{\varphi}}{\partial_3\widetilde{\varphi}}, \partial_3 v^i\right]$ is part of $C_i(v^i)$, which yields $-4\frac{\mathcal{T}^\gamma\widetilde{\mathbf{N}}_i}{\partial_3\widetilde{\varphi}}(\partial_3\mathcal{T}^\beta v^i)$ with $|\gamma| = 1, |\beta| = 3$, and so (6.21) contributes to

$$4\int_\Omega(\mathcal{T}^\alpha\check{q})(\mathcal{T}^\gamma\widetilde{\mathbf{N}}_i)(\partial_3\mathcal{T}^\beta v^i)\,\mathrm{d}x. \tag{6.22}$$

Note that this is straightforward to control if $\mathcal{T}^\alpha$ consists of at least one spatial derivative because of the reduction scheme on $\check{q}$. However, when $\mathcal{T}^\alpha = \partial_t^4$, we have

$$4\int_\Omega(\partial_t^4\check{q})(\partial_t\widetilde{\mathbf{N}}_i)(\partial_3\partial_t^3 v^i)\,\mathrm{d}x. \tag{6.23}$$

We can of course control this by $4\|\partial_t^4\check{q}\|_0\|\partial_t^3 v\|_1\|\partial_t\widetilde{\mathbf{N}}\|_\infty$ but this is not uniform in λ, as the reduction scheme on $\check{q}$ no longer verifies without spatial derivatives[9]. It turns out that (6.23) can be canceled by a term (at the top order) generated by the boundary integral $-\int_\Sigma \mathbf{Q}(\mathbf{V}\cdot\widetilde{N})\,\mathrm{d}x'$ after employing Gauss' theorem. This will be discussed in Subsection 6.3.3.

Since $\rho \geq 1$ and $\partial_3\widetilde{\varphi} \geq c_0 > 0$, we have $\int_\Omega \rho|\mathbf{V}|^2\partial_3\widetilde{\varphi}\,\mathrm{d}x \geq C\|\mathbf{V}\|_0^2$, and $\lambda^2\int_\Omega |\mathbf{Q}|^2\partial_3\widetilde{\varphi}\,\mathrm{d}x \geq C\lambda^2\|\mathbf{Q}\|_0^2$, for some universal positive C. Also, the definition of good unknowns indicates

$$\|\mathcal{T}^\alpha v\|_0 \leq \|\mathbf{V}\|_0 + \|\partial_3^{\widetilde{\varphi}} v\mathcal{T}^\alpha\widetilde{\varphi}\|_0, \quad \lambda\|\mathcal{T}^\alpha\check{q}\|_0 \leq \lambda\left(\|\mathbf{Q}\|_0 + \|\partial_3^{\widetilde{\varphi}}\check{q}\mathcal{T}^\alpha\widetilde{\varphi}\|_0\right),$$

and both $\|\partial_3^{\widetilde{\varphi}} v\mathcal{T}^\alpha\widetilde{\varphi}\|_0$ and $\lambda\|\partial_3^{\widetilde{\varphi}}\check{q}\mathcal{T}^\alpha\widetilde{\varphi}\|_0$ are of lower orders, which can be controlled by the RHS of (6.5).

6.3.2 Estimate for $-\int_\Sigma \mathbf{Q}(\mathbf{V}\cdot\widetilde{N})\,\mathrm{d}x'$ with at least one $\overline{\partial}$

We devote the remaining of this section to discuss $-\int_\Sigma \mathbf{Q}(\mathbf{V}\cdot\widetilde{N})\,\mathrm{d}x'$. We will highlight key estimates only in these notes, and we refer to [32] for the full treatment. Generally speaking, it is *much simpler* if $\mathcal{T}^\alpha$ contains at least one $\overline{\partial}$ compared to the case with time derivatives only. This is simply due to that one can integrate one-half spatial (tangential) derivatives by parts on the boundary if needed. Also, as mentioned above, (6.22) is straightforward to control.

Since $\mathfrak{D}^\alpha$ contains at least one $\overline{\partial}$ with $\alpha = 4$, we shall simply denote the mixed derivatives to be $\mathfrak{D}^3\overline{\partial}$. Invoking (6.17) and (6.18), $-\int_\Sigma \mathbf{Q}(\mathbf{V}\cdot\widetilde{N})\,\mathrm{d}x'$ contributes mainly to

$$ST_1 := \sigma\int_\Sigma \mathfrak{D}^3\overline{\partial}\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\widetilde{\psi}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}}\right)\partial_t\mathfrak{D}^3\overline{\partial}\psi\,\mathrm{d}x', \quad ST_2 := -\kappa^2\int_\Sigma \mathfrak{D}^3\overline{\partial}(1-\overline{\Delta})\partial_t\psi(\mathfrak{D}^3\overline{\partial}\partial_t\psi)\,\mathrm{d}x',$$

$$RT = \int_\Sigma \partial_3 q\mathfrak{D}^3\overline{\partial}\widetilde{\psi}\partial_t\mathfrak{D}^3\overline{\partial}\psi\,\mathrm{d}x'.$$

First, since $1-\overline{\Delta} = \langle\overline{\partial}\rangle^2$, where $\langle\cdot\rangle$ denotes the Japanese bracket,

$$ST_2 = -|\kappa\mathfrak{D}^3\overline{\partial}\partial_t\psi(t)|_1^2$$

which yields the κ-weighted L_t^2-energy norm after integrating in time. Second, similar to the analysis in Subsection 4.1, ST_1 has top-order contribution:

$$-\sigma\int_\Sigma\left(\frac{\mathfrak{D}^3\overline{\partial}\overline{\nabla}\widetilde{\psi}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}} - \frac{\overline{\nabla}\widetilde{\psi}\cdot\mathfrak{D}^3\overline{\partial}\overline{\nabla}\widetilde{\psi}}{\left(\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}\right)^3}\overline{\nabla}\widetilde{\psi}\right)\cdot\overline{\nabla}\mathfrak{D}^3\overline{\partial}\partial_t\psi\,\mathrm{d}x'. \tag{6.24}$$

Noticing that there is a mismatch between $\mathfrak{D}^3\overline{\partial}\overline{\nabla}\widetilde{\psi}$ and $\overline{\nabla}\mathfrak{D}^3\overline{\partial}\partial_t\psi$ results in the κ-boundary conditions. Since $\widetilde{\psi} := \Lambda_\kappa^2\psi$, we treat this by moving one Λ_κ from the former to the latter to create symmetry, which can be done through

$$|[\Lambda_\kappa, f]\overline{\partial}g|_0 \lesssim \kappa|f|_{W^{1,\infty}(\Sigma)}|\overline{\partial}g|_0. \tag{6.25}$$

Specifically speaking,

$$(6.24) = -\sigma\int_\Sigma \frac{(\mathfrak{D}^3\overline{\partial}\overline{\nabla}\Lambda_\kappa\psi)(\overline{\nabla}\mathfrak{D}^3\overline{\partial}\partial_t\Lambda_\kappa\psi)}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}} - \frac{(\overline{\nabla}\widetilde{\psi}\cdot\mathfrak{D}^3\overline{\partial}\overline{\nabla}\Lambda_\kappa\psi)(\overline{\nabla}\widetilde{\psi}\cdot\overline{\nabla}\mathfrak{D}^3\overline{\partial}\partial_t\Lambda_\kappa\psi)}{\left(\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}\right)^3}\,\mathrm{d}x'$$

$$-\sigma\int_\Sigma \mathfrak{D}^3\overline{\partial}\overline{\nabla}\Lambda_\kappa\psi\cdot\left(\left[\Lambda_\kappa, \frac{1}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}}\right]\overline{\nabla}\partial_t\mathfrak{D}^3\overline{\partial}\psi\right)\mathrm{d}x' + \sigma\int_\Sigma \mathfrak{D}^3\overline{\partial}\overline{\nabla}_i\Lambda_\kappa\psi\cdot\left(\left[\Lambda_\kappa, \frac{\overline{\nabla}_i\widetilde{\psi}\overline{\nabla}_j\widetilde{\psi}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}^3}\right]\overline{\nabla}_j\partial_t\mathfrak{D}^3\overline{\partial}\psi\right)\mathrm{d}x',$$

[9] See also the last paragraph of Subsection 6.2.

where terms in the first line are treated similarly to (4.4), while the time integral of terms on the second line requires (6.25) to control. Particularly, we put both $\frac{1}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}}$ and $\frac{\overline{\nabla}_i\widetilde{\psi}\overline{\nabla}_j\widetilde{\psi}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}^3}$ in $W^{1,\infty}$, attach the κ-weight with $|\overline{\partial}^2\mathfrak{D}^3\partial_t\psi|_0$ and then adapt Young's inequality to control it by the energy generated by ST_2 mentioned above.

Third, it suffices to consider

$$RT' := \int_\Sigma \partial_3 q \mathfrak{D}^3\overline{\partial}\Lambda_\kappa\psi\partial_t\mathfrak{D}^3\overline{\partial}\Lambda_\kappa\psi \,\mathrm{d}x' \tag{6.26}$$

after moving one Λ_κ from $\widetilde{\psi}$ to ψ, similar as above. It can be seen that the control of RT' requires

$$|\partial_t\mathfrak{D}^3\overline{\partial}\Lambda_\kappa\psi|_0,$$

which can be controlled by the tangential energy in Theorem 6.3 *but not uniform in σ*. Nevertheless, RT' contributes to

$$-\frac{d}{dt}\int_\Sigma(-\partial_3 q)(\mathfrak{D}^3\overline{\partial}\Lambda_\kappa\psi)^2\,\mathrm{d}x' \tag{6.27}$$

provided with the Rayleigh-Taylor sign condition $-\partial_3 q \geq c_0 > 0$. In conclusion, it holds that:

Despite the control of RT' failing to be σ-uniform, it yields a positive energy term provided the Rayleigh-Taylor sign condition.

6.3.3 Estimate for $-\int_\Sigma \mathbf{Q}(\mathbf{V}\cdot\widetilde{N})\,\mathrm{d}x'$ with time derivatives only

The terms corresponding to ST_1 and ST_2 above are treated similarly, whereas

$$RT^* = \int_\Sigma \partial_3 q\partial_t^4\widetilde{\psi}\partial_t^5\psi\,\mathrm{d}x'$$

contributes also to an energy term provided the Rayleigh-Taylor sign condition holds after moving one Λ_κ from $\partial_t^4\widetilde{\psi}$ to $\partial_t^5\psi$ via (6.25). However, in the case without the sign condition, we have no direct control of $|\partial_t^5\psi|_0$ as the surface tension energy has to have at least one $\overline{\partial}$. We need to treat this case alternatively by writing

$$\partial_t^5\psi = -(\overline{v}\cdot\overline{\partial})\partial_t^4\psi + \partial_t^4 v\cdot\widetilde{N} - [\partial_t^4, \overline{v}, \overline{\nabla}\widetilde{\psi}],$$

and then transform some terms in RT^* as interior integrals on Ω by Gauss' theorem.

On the other hand, as mentioned previously in Subsection 6.3.1, in the case with full-time derivatives, i.e., $\mathcal{T}^\alpha = \partial_t^4$, dedicated cancellations are required to obtain a uniform-in-λ energy estimate. It turns out that this cancellation is crucial to get rid of some top-order boundary terms when studying this boundary integral. In particular, we study a term generated by

$$\int_\Sigma \mathbf{Q}S_1\,\mathrm{d}x',$$

where S_1 comes from (6.18). This yields a top-order term on Σ:

$$ST_3 := -4\int_\Sigma \partial_t^4\check{q}\partial_t^3 v^i\partial_t\widetilde{N}_i\,\mathrm{d}x'. \tag{6.28}$$

Invoking the Gauss' theorem, we have

$$-\int_\Sigma \partial_t^4\check{q}\partial_t^3 v^i\partial_t\widetilde{\mathbf{N}}_i\,\mathrm{d}x' = -\int_\Omega \partial_3(\partial_t^4\check{q}\partial_t^3 v^i\partial_t\widetilde{\mathbf{N}}_i)\,\mathrm{d}x.$$

This holds since ∂_3 is the flat normal derivative (with respect to $\Sigma\cup\Sigma_b$) and $\partial_t\widetilde{\mathbf{N}} = \partial_t e_3 = 0$ on Σ_b. Now,

$$ST_3 = -4\int_\Omega \partial_t^4\check{q}\partial_3\partial_t^3 v^i\partial_t\widetilde{\mathbf{N}}\,\mathrm{d}x - 4\int_\Omega \partial_t^4\partial_3\check{q}\partial_t^3 v^i\partial_t\widetilde{\mathbf{N}}\,\mathrm{d}x - 4\int_\Omega \partial_t^4\check{q}\partial_t^3 v^i\partial_t\partial_3\widetilde{\mathbf{N}}\,\mathrm{d}x.$$

The first term cancels with (6.23), and we treat the second and third terms under the time integral as

$$\int_0^T\int_\Omega \partial_t^4\partial_3\check{q}\partial_t^3 v^i\partial_t\widetilde{\mathbf{N}}\,\mathrm{d}x = \int_\Omega \partial_t^3\partial_3\check{q}\partial_t^3 v^i\partial_t\widetilde{\mathbf{N}}\,\mathrm{d}x - \int_0^T\int_\Omega \partial_t^3\partial_3\check{q}\partial_t(\partial_t^3 v^i\partial_t\widetilde{\mathbf{N}})\,\mathrm{d}x,$$
$$\int_0^T\int_\Omega \partial_t^4\check{q}\partial_t^3 v^i\partial_t\partial_3\widetilde{\mathbf{N}}\,\mathrm{d}x = \int_\Omega \partial_t^3\check{q}\partial_t^3 v^i\partial_t\partial_3\widetilde{\mathbf{N}}\,\mathrm{d}x - \int_0^T\int_\Omega \partial_t^3\check{q}\partial_t(\partial_t^3 v^i\partial_t\partial_3\widetilde{\mathbf{N}})\,\mathrm{d}x\,\mathrm{d}x.$$

Here, we treat the terms without time integral using Young's inequality, i.e.,

$$\int_\Omega \partial_t^3\partial_3\check{q}\partial_t^3 v^i\partial_t\widetilde{\mathbf{N}}\,\mathrm{d}x + \int_\Omega \partial_t^3\check{q}\partial_t^3 v^i\partial_t\partial_3\widetilde{\mathbf{N}}\,\mathrm{d}x \lesssim \epsilon\|\partial_t^3\check{q}\|_1^2 + C(\epsilon^{-1})\|\partial_t^3 v\|_0^2\|\partial_t\widetilde{\mathbf{N}}\|_\infty^2,$$

and the last term is of lower orders and can be controlled by $\mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t$.

7 Major ideas of the proof III: Solution of the linearized κ-equations

The final step of showing the LWP for (3.6) is to prove that the approximate κ-equations (5.1) admits a unique solution *for each fixed* $\kappa > 0$ in $[0, T_\kappa]$.

7.1 Linearization

We construct the solution of (5.1) by iterating the solutions of the *linearized* version of these equations. Particularly, let

$$(\psi^{(0)}, v^{(0)}, \rho^{(0)}) := (0, \mathbf{0}, 1), \quad \text{and } \psi^{(-1)} = 0.$$

Assume that $(\psi^{(k)}, v^{(k)}, \rho^{(k)})$ is known for all $k \le n$.

Notation 7.1 (Linearized derivatives)**.** We define:

i. (Linearized Eulerian time derivative) $\partial_t^{\widetilde{\varphi}^{(n)}} := \partial_t - (\partial_t\varphi^{(n)})(\partial_3\widetilde{\varphi}^{(n)})^{-1}\partial_3$.
ii. (Linearized Eulerian spatial derivatives) $\nabla_\tau^{\widetilde{\varphi}^{(n)}} = \partial_\tau^{\widetilde{\varphi}^{(n)}} := \partial_\tau - (\partial_\tau\widetilde{\varphi}^{(n)})(\partial_3\widetilde{\varphi}^{(n)})^{-1}\partial_3$, $\tau = 1, 2$, and $\nabla_3^{\widetilde{\varphi}^{(n)}} = \partial_3^{\widetilde{\varphi}^{(n)}} := (\partial_3\widetilde{\varphi}^{(n)})^{-1}\partial_3$.
iii. (Linearized material derivative) $D_t^{\widetilde{\varphi}^{(n)}} := \partial_t + \overline{v}^{(n)}\cdot\overline{\partial} + (\partial_3\widetilde{\varphi}^{(n)})^{-1}(v^{(n)}\cdot\widetilde{\mathbf{N}}^{(n-1)} - \partial_t\varphi^{(n)})\partial_3$.

We are going to show the existence $(\psi^{(n+1)}, v^{(n+1)}, \rho^{(n+1)})$, defined to be the solution of the following linear system:

$$\begin{cases}
\rho^{(n)}D_t^{\widetilde{\varphi}^{(n)}}v^{(n+1)} + \nabla^{\widetilde{\varphi}^{(n)}}\check{q}^{(n+1)} = -(\rho^{(n)} - 1)ge_3, & \text{in } [0,T]\times\Omega,\\
\lambda^2 D_t^{\widetilde{\varphi}^{(n)}}\check{q}^{(n+1)} + \nabla^{\widetilde{\varphi}^{(n)}}\cdot v^{(n+1)} = g\lambda^2 v_3^{(n)}, & \text{in } [0,T]\times\Omega,\\
\partial_t\psi^{(n+1)} = v^{(n+1)}\cdot\widetilde{N}^{(n)}, & \text{on } [0,T]\times\Sigma,\\
\check{q}^{(n+1)} = g\widetilde{\psi}^{(n)} - \sigma\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\widetilde{\psi}^{(n)}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}^{(n)}|^2}}\right) + \kappa^2(1-\overline{\Delta})\partial_t\psi^{(n+1)}, & \text{on } [0,T]\times\Sigma,\\
v_3^{(n+1)} = 0, & \text{on } [0,T]\times\Sigma_b,\\
(\psi^{(n+1)}, v^{(n+1)}, \check{q}^{(n+1)})|_{t=0} = (\psi_{\kappa,0}, v_{\kappa,0}, \check{q}_{\kappa,0}),
\end{cases} \tag{7.1}$$

where $\check{q}^{(n+1)} := q(\rho^{(n+1)}) + g\widetilde{\varphi}^{(n)}$. For the sake of simplicity, we denote $(\psi^{(n+1)}, v^{(n+1)}, \rho^{(n+1)}, \check{q}^{(n+1)}) = (\psi, v, \rho, \check{q})$, $(\psi^{(n)}, v^{(n)}, \rho^{(n)}) = (\mathring{\psi}, \mathring{v}, \mathring{\rho})$, $\varphi^{(n)} = \mathring{\varphi}$, and $\psi^{(n-1)} = \dot{\psi}$, in the rest of this section. Given these,

i. $\partial_t^{\widetilde{\mathring{\varphi}}} := \partial_t - (\partial_t\mathring{\varphi})(\partial_3\widetilde{\mathring{\varphi}})^{-1}\partial_3$.
ii. $\nabla_\tau^{\widetilde{\mathring{\varphi}}} = \partial_\tau^{\widetilde{\mathring{\varphi}}} := \partial_\tau - (\partial_\tau\widetilde{\mathring{\varphi}})(\partial_3\widetilde{\mathring{\varphi}})^{-1}\partial_3$, $\tau = 1, 2$, and $\nabla_3^{\widetilde{\mathring{\varphi}}} = \partial_3^{\widetilde{\mathring{\varphi}}} := (\partial_3\widetilde{\mathring{\varphi}})^{-1}\partial_3$.
iii. $D_t^{\widetilde{\mathring{\varphi}}} := \partial_t + \overline{\mathring{v}}\cdot\overline{\partial} + (\partial_3\widetilde{\mathring{\varphi}})^{-1}(\mathring{v}\cdot\widetilde{\dot{\mathbf{N}}} - \partial_t\mathring{\varphi})\partial_3$, where $\widetilde{\dot{\mathbf{N}}} = (-\partial_1\widetilde{\dot{\psi}}, -\partial_2\widetilde{\dot{\psi}}, 1)$.

Also, system (7.1) becomes

$$
\begin{cases}
\mathring{\rho} D_t^{\mathring{\widetilde{\varphi}}} v + \nabla^{\mathring{\widetilde{\varphi}}} \check{q} = -(\mathring{\rho} - 1) g e_3, & \text{in } [0,T] \times \Omega, \\
\lambda^2 D_t^{\mathring{\widetilde{\varphi}}} \check{q} + \nabla^{\mathring{\widetilde{\varphi}}} \cdot v = g \lambda^2 \mathring{v}^3, & \text{in } [0,T] \times \Omega, \\
\partial_t \psi = v \cdot \mathring{\widetilde{N}}, & \text{on } [0,T] \times \Sigma, \\
\check{q} = g \mathring{\widetilde{\psi}} - \sigma \overline{\nabla} \cdot \left(\frac{\overline{\nabla} \mathring{\widetilde{\psi}}}{\sqrt{1 + |\overline{\nabla} \mathring{\widetilde{\psi}}|^2}} \right) + \kappa^2 (1 - \overline{\Delta}) \partial_t \psi, & \text{on } [0,T] \times \Sigma, \\
v^3 = 0, & \text{on } [0,T] \times \Sigma_b, \\
(\psi, v, \check{q})|_{t=0} = (\psi_{\kappa,0}, v_{\kappa,0}, \check{q}_{\kappa,0}).
\end{cases}
\tag{7.2}
$$

Remark. Note that the linearized material derivative $D_t^{\mathring{\widetilde{\varphi}}}$ is *not* $\partial^{\mathring{\widetilde{\varphi}}} + \mathring{v} \cdot \nabla^{\mathring{\widetilde{\varphi}}}$! Instead, we linearize $D_t^{\widetilde{\varphi}}$ directly from $\partial_t + \overline{v} \cdot \overline{\partial} + (v \cdot \widetilde{\mathbf{N}} - \partial_t \varphi) \partial_3^{\widetilde{\varphi}}$ by replacing $v \cdot \widetilde{\mathbf{N}} - \partial_t \varphi$ as $\mathring{v} \cdot \dot{\widetilde{\mathbf{N}}} - \partial_t \mathring{\varphi}$, where the latter matches with the former on Σ in view of the the linearized kinematic condition. This design is important since the linearized transport theorem would yield top-order boundary terms otherwise.

Remark. The surface tension $\sigma \mathcal{H}(\overline{\nabla} \mathring{\widetilde{\psi}}, \overline{\nabla} \mathring{\widetilde{\psi}}) = -\sigma \overline{\nabla} \cdot \left(\frac{\overline{\nabla} \mathring{\widetilde{\psi}}}{\sqrt{1 + |\overline{\nabla} \mathring{\widetilde{\psi}}|^2}} \right)$ is replaced by a given function in (7.2). However, since κ is fixed, we can still control ψ from the artificial viscosity.

7.2 Solution of the linearized equations in L^2

Next, we show the linearized equations (7.2) admits a solution. We could achieve this by employing Galerkin's method if Ω was bounded. Also, Galerkin's method may also be adapted while Ω is unbounded by approximating Ω by $\Omega_r := \Omega \cap B(0, r)$ as $r \to \infty$. Here, however, we consider (7.2) as a symmetric hyperbolic system with characteristic boundary conditions[10], which then allows us to adapt the result of Lax and Phillips [27] to prove the existence.

However, we have to rewrite (7.2) with *homogeneous boundary conditions* before adapting [27]'s arguments. Our strategy to achieve this is by first setting $\mathring{\mathfrak{h}}$ to be the harmonic extension of $g \mathring{\widetilde{\psi}} - \sigma \overline{\nabla} \cdot \left(\frac{\overline{\nabla} \mathring{\widetilde{\psi}}}{\sqrt{1 + |\overline{\nabla} \mathring{\widetilde{\psi}}|^2}} \right)$, i.e., setting

$$
\begin{cases}
-\Delta \mathring{\mathfrak{h}} = 0, & \text{in } \Omega, \\
\mathring{\mathfrak{h}} = g \mathring{\widetilde{\psi}} - \sigma \overline{\nabla} \cdot \left(\frac{\overline{\nabla} \mathring{\widetilde{\psi}}}{\sqrt{1 + |\overline{\nabla} \mathring{\widetilde{\psi}}|^2}} \right) & \text{on } \Sigma, \\
\partial_3 \mathring{\mathfrak{h}} = 0, & \text{on } \Sigma_b,
\end{cases}
$$

and $\underline{q} := \check{q} - \mathring{\mathfrak{h}}$. Then replacing $\partial_t \psi$ by $v \cdot \mathring{\widetilde{N}}$ in the linearized artificial viscosity. Thus,

$$
\underline{q} = \kappa^2 (1 - \overline{\Delta})(v \cdot \mathring{\widetilde{N}}), \quad \text{on } \Sigma.
$$

Re-expressing (7.2) in terms of $\underline{q}$, we have

$$
\begin{cases}
\mathring{\rho} D_t^{\mathring{\widetilde{\varphi}}} v + \nabla^{\mathring{\widetilde{\varphi}}} \underline{q} = -\nabla^{\mathring{\widetilde{\varphi}}} \mathring{\mathfrak{h}} - (\mathring{\rho} - 1) g e_3, & \text{in } [0,T] \times \Omega, \\
\lambda^2 D_t^{\mathring{\widetilde{\varphi}}} \underline{q} + \nabla^{\mathring{\widetilde{\varphi}}} \cdot v = \lambda^2 (g \mathring{v}^3 - D_t^{\mathring{\widetilde{\varphi}}} \mathring{\mathfrak{h}}), & \text{in } [0,T] \times \Omega, \\
\underline{q} = q(\rho) + g \mathring{\widetilde{\varphi}} - \mathring{\mathfrak{h}}, & \text{in } [0,T] \times \Omega, \\
\underline{q} = \kappa^2 (1 - \overline{\Delta})(v \cdot \mathring{\widetilde{N}}), & \text{on } [0,T] \times \Sigma, \\
v_3 = 0, & \text{on } [0,T] \times \Sigma_b, \\
(v, \rho)|_{t=0} = (v_{\kappa,0}, \rho_{\kappa,0}).
\end{cases}
\tag{7.3}
$$

Let

$$
U := (\underline{q}, v_1, v_2, v_3)^T.
$$

[10] Since $\rho = 0$ on the free surface boundary, the compressible Euler equations modeling a *gas* are not hyperbolic on the moving surface boundary.

Then (7.3) turns into

$$A_0 \partial_t U + \sum_{i=1}^{3} A_i \partial_i U = \mathring{f}, \tag{7.4}$$

where

$$\mathring{f} = \left(\lambda^2 (g\mathring{v}^3 - D_t^{\tilde{\mathring{\varphi}}} \mathring{\mathfrak{h}}), -\partial_1^{\tilde{\mathring{\varphi}}} \mathring{\mathfrak{h}}, -\partial_2^{\tilde{\mathring{\varphi}}} \mathring{\mathfrak{h}}, -\partial_3^{\tilde{\mathring{\varphi}}} \mathring{\mathfrak{h}} - g(\mathring{\rho} - 1)\right)^T, \quad A_0 = A_0(\mathring{U}) = \operatorname{diag}(\lambda^2, \mathring{\rho}, \mathring{\rho}, \mathring{\rho}),$$

$$A_j = A_j(\mathring{U}) = \begin{pmatrix} \lambda^2 \mathring{v}_i & e_i^T \\ e_i & \mathring{\rho}\mathring{v}_i \mathbf{I}_3 \end{pmatrix}, \quad j = 1, 2, \quad \text{and} \quad A_3 = A_3(\mathring{U}) = \begin{pmatrix} \lambda^2 (\mathring{v} \cdot \tilde{\mathring{\mathbf{N}}} - \partial_t \mathring{\varphi}) & \tilde{\mathring{\mathbf{N}}}^T \\ \tilde{\mathring{\mathbf{N}}} & \mathring{\rho}(\mathring{v} \cdot \tilde{\mathring{\mathbf{N}}} - \partial_t \mathring{\varphi}) \mathbf{I}_3. \end{pmatrix}$$

Now, the result of [27] indicates that (7.4) admits a solution in some functional space $\mathbb{X}$ if one can:

I. Prove a priori energy estimate of (7.4) in $\mathbb{X}$.
II. Prove a priori energy estimate of the dual system of (7.4) in $\mathbb{X}^*$.

Writing (7.4) as $LU = A^\mu \partial_\mu U = \mathring{f}$, $0 \le \mu \le 3$, then its dual system is given by

$$L^* U^* = \partial_\mu (A^\mu U^*) = \mathring{f}^*, \quad U^* := (\underline{q}^*, v_1^*, v_2^*, v_3^*)^T. \tag{7.5}$$

We shall choose $\mathbb{X} = L^2([0, T] \times \Omega)$ since $L^2([0, T] \times \Omega)^* = L^2([0, T] \times \Omega)$.

Testing the dual system (7.5) with U^*, it is straightforward to control all the interior integrals. However, there is an issue with the boundary integral. Note that from $\underline{q} = \kappa^2 (1 - \overline{\Delta})(v \cdot \tilde{\mathring{N}})$ we derive $\underline{q}^* = \kappa^2 (\overline{\Delta} - 1)(v^* \cdot \tilde{\mathring{N}})$, and thus the boundary integral

$$\begin{aligned} -\int_\Sigma (U^*)^T A_3 U^* \, \mathrm{d}x' &= -2 \int_\Sigma \underline{q}^* (v^* \cdot \tilde{\mathring{N}}) \, \mathrm{d}x' \\ = 2\kappa^2 \int_\Sigma (v^* \cdot \tilde{\mathring{N}}) \Big((1 - \overline{\Delta})(v^* \cdot \tilde{\mathring{N}})\Big) \, \mathrm{d}x' &= 2\kappa^2 |v^* \cdot \tilde{\mathring{N}}|_1^2 \end{aligned}$$

does not contribute to a positive term on the LHS. To resolve this issue, we introduce an extra μ-viscosity to $\underline{q}$ on Σ, so that

$$\underline{q}|_\Sigma = \kappa^2 (1 - \overline{\Delta})(v \cdot \tilde{\mathring{N}}) + \mu (1 - \overline{\Delta}) \partial_t (v \cdot \tilde{\mathring{N}}), \tag{7.6}$$

and so

$$\underline{q}^*|_\Sigma = \kappa^2 (\overline{\Delta} - 1)(v^* \cdot \tilde{\mathring{N}}) + \mu (1 - \overline{\Delta}) \partial_t (v^* \cdot \tilde{\mathring{N}}). \tag{7.7}$$

In light of (7.6), we have

$$\begin{aligned} -\int_\Sigma U^T A_3 U \, \mathrm{d}x = -2 \int_\Sigma (v \cdot \widetilde{N}) \underline{q} \, \mathrm{d}x' &= -2\kappa^2 \int_\Sigma (v \cdot \tilde{\mathring{N}})(1 - \overline{\Delta})(v \cdot \tilde{\mathring{N}}) \, \mathrm{d}x' - 2\mu \int_\Sigma (v \cdot \tilde{\mathring{N}}) \partial_t \Big((1 - \overline{\Delta})(v \cdot \tilde{\mathring{N}})\Big) \, \mathrm{d}x' \\ &= -2\kappa^2 |v \cdot \tilde{\mathring{N}}|_1^2 - \mu \frac{d}{dt} |v \cdot \tilde{\mathring{N}}|_1^2. \end{aligned}$$

The terms on the second line are all of the correct sign and give the control of $|v \cdot \tilde{\mathring{N}}|_1^2$ after moving to the LHS and then integrating in time. In addition, by the virtue of (7.7), we have

$$\begin{aligned} -\int_\Sigma (U^*)^T A_3 U^* \, \mathrm{d}x = -2 \int_\Sigma (v^* \cdot \widetilde{N}) \underline{q}^* \, \mathrm{d}x' &= 2\kappa^2 \int_\Sigma (v^* \cdot \tilde{\mathring{N}})(1 - \overline{\Delta})(v^* \cdot \tilde{\mathring{N}}) \, \mathrm{d}x' - 2\mu \int_\Sigma (v^* \cdot \tilde{\mathring{N}}) \partial_t \Big((1 - \overline{\Delta})(v^* \cdot \tilde{\mathring{N}})\Big) \, \mathrm{d}x' \\ &= 2\kappa^2 |v^* \cdot \tilde{\mathring{N}}|_1^2 - \mu \frac{d}{dt} |v^* \cdot \tilde{\mathring{N}}|_1^2. \end{aligned}$$

Similar to above, $-\mu \frac{d}{dt} |v^* \cdot \tilde{\mathring{N}}|_1^2$ contributes to $\mu |v^* \cdot \tilde{\mathring{N}}|_1^2$ after moving to the LHS and then integrating in time, and this gives the control of $2\kappa^2 |v^* \cdot \tilde{\mathring{N}}|_1^2$ up to a constant depends on μ^{-1}.

In conclusion, we can close the $L^2(\Omega)$ energy estimate for both (7.4) and its dual system for each fixed $\mu > 0$. Thus, from [27] we deduce that (7.4) admits a *weak* solution in $L^2(\Omega)$ equipped with the boundary condition (7.6). Also, since the L^2-energy estimate of (7.4) is uniform in μ, we can pass $\mu \to 0$ and then obtain a weak solution of (7.3). This weak solution is in fact a strong solution thanks to the arguments in [34, Section 2.2.3] and [37, Theorem 4&8]. Lastly, we recover ψ by defining it through

$$\partial_t \psi = v \cdot \mathring{\widetilde{N}}, \quad \text{on} \quad \Sigma,$$

to obtain a solution of (7.2).

7.3 A sketch of the Picard iteration scheme

With the L^2-solution $(\psi, v, \check{q})$ of (7.2) constructed above, we can show that $(\psi, v, \check{q}) \in H^5(\Sigma) \times H^4(\Omega) \times H^4(\Omega)$ via the energy estimate. This is conducted similarly to the energy estimate of the nonlinear κ-equations in Section 6. We use the following Hodge-type div-curl estimate in [5]:

$$\|X\|^2_{H^s(\Omega)} \le C(|\mathring{\widetilde{\psi}}|^2_{s+\frac{1}{2}}) \left(\|X\|_0^2 + \|\nabla^{\mathring{\widetilde{\varphi}}} \cdot X\|^2_{s-1} + \|\nabla^{\mathring{\widetilde{\varphi}}} \times X\|^2_{s-1} + |X \cdot \mathring{\widetilde{N}}|^2_{s-\frac{1}{2}} + |X \cdot e_3|^2_{H^{s-\frac{1}{2}}(\Sigma_b)} \right). \tag{7.8}$$

Note that $\|\mathring{\widetilde{\psi}}\|_{s+\frac{1}{2}} \lesssim_{\kappa^{-1}} \|\mathring{\psi}\|_s$, which can be used here as $\kappa > 0$ is fixed. Let $X = \partial_t^k v$ and $s = 4 - k$, $k = 0, 1, 2, 3$, and the last term of (7.8) is 0. Since $|\partial_t^k v \cdot \mathring{\widetilde{N}}|_{3.5-k} \le \widetilde{C} |\partial_t^k (v \cdot \mathring{\widetilde{N}})|_{3.5-k}$, where $\widetilde{C}$ depends on $\mathring{\psi}$ (which is given), we control $|\partial_t^k (v \cdot \mathring{\widetilde{N}})|_{3.5-k} \le |\langle \overline{\partial} \rangle^{-1} \partial_t^k (v \cdot \mathring{\widetilde{N}})|_{4-k}$ by applying the elliptic estimate to

$$(1 - \overline{\Delta})(v \cdot \mathring{\widetilde{N}}) = \check{q} - g\mathring{\widetilde{\psi}} + \sigma \overline{\nabla} \cdot \left(\frac{\overline{\nabla} \mathring{\widetilde{\psi}}}{\sqrt{1 + |\overline{\nabla} \mathring{\widetilde{\psi}}|^2}} \right),$$

which is derived from the boundary condition of (7.2). On the other hand, by applying the arguments in Subsection 6.2, we can reduce the spatial derivatives of $\check{q}$ to tangential derivatives of v, and together with $\nabla^{\mathring{\widetilde{\varphi}}} \cdot v = \lambda^2 D_t^{\mathring{\widetilde{\varphi}}} \check{q} + g\lambda^2 \mathring{v}^3$, we can eventually reduce the control of $\|\partial_t^k v\|_{4-k}$ and $\|\partial_t^k \check{q}\|_{4-k}$, $k = 0, 1, 2, 3$, to $\lambda \|\partial_t^4 \check{q}\|_0$ and $\|\mathcal{T}^4 v\|_0$ (apart from the curl part that we have no problem to control), which can be estimated similar to Subsection 6.3.

We next iterate the linearized solution to achieve a solution of the nonlinear κ-system (5.1). This is done via studying the difference between linearized equations of $(\psi^{(n+1)}, v^{(n+1)}, \rho^{(n+1)})$ and $(\psi^{(n)}, v^{(n)}, \rho^{(n)})$. We omit the details. We however point out that we can study the difference between two systems in a lower-order Sobolev space, say $v^{(n+1)} - v^{(n)}, \rho^{(n+1)} - \rho^{(n)} \in H^3(\Omega)$, and $\psi^{(n+1)} - \psi^{(n)} \in H^4(\Sigma)$.

Thanks to uniform-in-κ Theorem 6.1, we can pass the sequence of κ-solutions to the limit (possibly after passing to a subsequence) that solves (3.6) in $[0, T]$. In addition, the uniqueness of this solution follows from the energy estimate of the difference between two successive systems of (3.6) with 0 initial data.

8 Remarks on the incompressible and zero surface tension limits with weaker initial data

Recall that in Subsection 6.2 we discussed the λ-weight distribution on the energies in Theorem 6.1. In fact, we can still prove the incompressible and zero surface tension double limits with a weaker energy functional. Particularly, since the energy norms generated by the continuity equation carry λ-weight, we do not require $\|\partial_t^{2+\ell} \check{q}_0\|^2_{2-\ell}$, $\quad 0 \le \ell \le 2$ to be uniformly bounded; instead, we require merely the boundedness of

$$\|\lambda^{s+1} \partial_t^{2+s} \check{q}_0\|^2_{2-s}, \quad 0 \le s \le 2.$$

Moreover, by the arguments in Subsection 6.2, for $1 \le \ell \le 2$ the $H^{2-\ell}(\Omega)$-norm of $\partial_t^{2+\ell} v$ must also be λ^ℓ-weighted. However, we do require $\|\partial_t \check{q}(0)\|_3$ to be bounded as we need $\|\partial_t q\|_3 \le \|\partial_t \check{q}\|_3 + \|g \partial_t \varphi\|_3$ to propagate the Rayleigh-Taylor sign condition.

Under these criteria, we can determine the λ-weight on each term of the new energy by

$$\begin{tikzcd}
 & \|\partial_t^2 v^{\lambda,\sigma}\|_2 \arrow[d, "\text{energy}"] & \|\lambda\partial_t^3 v^{\lambda,\sigma}\|_1 \arrow[d, "\text{energy}"] & \|\lambda^2\partial_t^4 v^{\lambda,\sigma}\|_0 \arrow[d, "\text{energy}"] \\
\|\partial_t \check{q}^{\lambda,\sigma}\|_3 \arrow[ur, "\text{reduction}"] & \|\lambda\partial_t^2 \check{q}^{\lambda,\sigma}\|_2 \arrow[ur, "\text{reduction}"] & \|\lambda^2\partial_t^3 \check{q}^{\lambda,\sigma}\|_1 \arrow[ur, "\text{reduction}"] & \|\lambda^3\partial_t^4 \check{q}^{\lambda,\sigma}\|_0
\end{tikzcd} \tag{8.1}$$

In consequence, weaker energy is given by

$$\begin{aligned} E^{weak}(t) = \sum_{k=0}^{1} \|\partial_t^k v(t)\|_{4-k}^2 + \|\partial^{1-k}\partial_t^k \check{q}(t)\|_3^2 + \sum_{k=0}^{1} |\sqrt{\sigma}\overline{\nabla}\partial_t^k \psi(t)|_{4-k}^2 + |\partial_t^k \psi(t)|_{4-k}^2 \\ + \sum_{s=0}^{2} \|\lambda^s \partial_t^{2+s} v^{\lambda,\sigma}(t)\|_{2-s}^2 + \|\lambda^{s+1}\partial_t^{2+s}\check{q}^{\lambda,\sigma}(t)\|_{2-s}^2 + \sum_{s=0}^{2} |\sqrt{\sigma}\lambda^s \overline{\nabla}\partial_t^{2+s}\psi^{\lambda,\sigma}(t)|_{2-s}^2 + |\lambda^s \partial_t^{2+s}\psi^{\lambda,\sigma}(t)|_{2-s}^2. \end{aligned} \tag{8.2}$$

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