Math 6261 23-03-03.

§3.1 Weak convergence.

- Let $\mu_n, \mu \in \text{Prob}(\mathbb{R})$. We say that $\mu_n \xrightarrow{w} \mu$ if

$$\int h \, d\mu_n \to \int h \, d\mu \quad \text{for all } h \in C_b(\mathbb{R}).$$

(the collection of bdd cts functions on $\mathbb{R}$)

- Equivalently,

$$\mu_n \xrightarrow{w} \mu \quad \text{if} \quad \lim_{n\to\infty} \mu_n(-\infty, x] = \mu(-\infty, x] \text{ for any non-atom } x \text{ of } \mu.$$

- The def of weak convergence of measures passes to that of distribution functions and r.v.'s in natural ways.

- (Helly-Bray thm) Let (F_n) be a sequence of DFs on $\mathbb{R}$. Then $\exists$ a subsequence (F_{n_k}) and a right cts function F on $\mathbb{R}$ such that

$0 \leq F \leq 1$ and

$$\lim_{k\to\infty} F_{n_k}(x) = F(x)$$

for every point x of cty of F.

Remark: The above F may not be a DF, since it may happen that $\lim_{x\to-\infty} F(x) \neq 0$ or $\lim_{x\to+\infty} F(x) \neq 1$.

Pf. Let $\bar{\mathbb{R}} = [-\infty, \infty]$. Then $\mathrm{Prob}(\bar{\mathbb{R}})$ is compact endowed with the weak topology.

Notice that $\mathrm{Prob}(\mathbb{R})$ can be viewed as a subspace of $\mathrm{Prob}(\bar{\mathbb{R}})$. So for any $(\mu_n) \subset \mathrm{Prob}(\mathbb{R})$, $\exists$ a subsequence (μ_{n_k}) and $\mu \in \mathrm{Prob}(\bar{\mathbb{R}})$ such that

$$\mu_{n_k} \xrightarrow{w} \mu.$$

Let $F(z) = \mu([-\infty, z])$ for $z \in \mathbb{R}$. Then

$$F_{n_k}(z) \to F(z) \quad \text{at the pt } z \text{ of cty of } F,$$

where F_{n_k} is the DF corresponding to μ_{n_k}. This proves Helly-Bray's Thm. $\blacksquare$

Q: When is the function F in Helly-Bray's Thm a DF?

According to the above proof, F is a DF $\Leftrightarrow$ $\mu \in \mathrm{Prob}(\mathbb{R})$

$\Leftrightarrow$ $\mu\{+\infty, -\infty\} = 0$.

Def (Tightness) A sequence $(\mu_n)_{n=1}^{\infty} \subset \mathrm{Prob}(\mathbb{R})$ is said to be tight if for each $\varepsilon > 0$, there exists $K > 0$ such that

$$\mu_n[-K, K] > 1 - \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

Similarly, a sequence (F_n) of DFs on $\mathbb{R}$ is said to be tight if for every $\varepsilon > 0$, there exists $K > 0$ such that

$$F_n(K) - F_n(-K-) > 1 - \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

Lem 3.6. Let (F_n) be a sequence of DFs on $\mathbb{R}$.

(1) If $F_n \xrightarrow{w} F$ for some DF F, then (F_n) is tight.

(2) If (F_n) is tight, then $\exists$ a subsequence (F_{n_k}) and a DF F such that

$$F_{n_k} \xrightarrow{w} F.$$

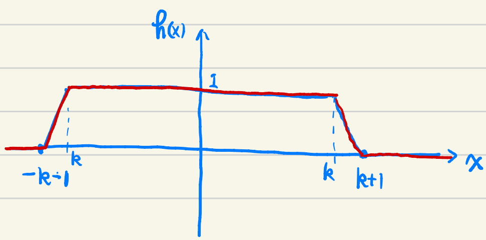
Pf. Here we only prove (1), and leave the proof of (2) as an exercise.

Let μ_n, μ be the prob. measures on $\mathbb{R}$ corresponding to F_n and F. Let $\varepsilon > 0$. Then $\exists\, k \in \mathbb{N}$ such that

$$\mu([-k, k]) > 1 - \varepsilon.$$

Take $h \in C_b(\mathbb{R})$ such that

$$h(x) = \begin{cases} 1 & \text{if } |x| \le k \\ \ge 0 & \text{if } k < |x| < k+1 \\ 0 & \text{if } |x| > k+1. \end{cases}$$



Then $\mu_n([-k-1, k+1]) \geq \int h \, d\mu_n \to \int h \, d\mu \geq \mu([-k,k])$.

It follow that $\exists$ N such that

$$\mu_n([-k-1, k+1]) \geq \mu([-k,k]) > 1-\varepsilon \qquad (1)$$

for all $n \geq N$.

Now pick $K > k+1$ such that

$$\mu_n([-K, K]) > 1-\varepsilon \quad \text{for} \quad n = 1, \cdots, N.$$

By (1) this implies that $\mu_n([-K,K]) > 1-\varepsilon$ for all $n \in \mathbb{N}$. $\square$

§3.2 Charateristic functions (CF)

Def. Let X be a r.v. The characteristic function of X is a map

$$\varphi : \mathbb{R} \to \mathbb{C}$$

defined by

$$\varphi(\theta) = E(e^{i\theta X})$$

$$= \int_{\mathbb{R}} e^{i\theta x} \, d\mu(x) \qquad (\mu \text{ is the law of } X).$$

$$= \int_{\mathbb{R}} \cos(\theta x) \, d\mu(x) + i \int_{\mathbb{R}} \sin(\theta x) \, d\mu(x).$$

Remark: We often write φ_X, φ_F, φ_μ for φ.

In analysis, φ is called the Fourier transform of μ.

or the Fourier-Stieltjes transform of F.

Lem 3.7. Let $\varphi = \varphi_X$ be the CF of a r.v. X. Then

① $\varphi(0) = 1$.

② $|\varphi(\theta)| \le 1$ for all θ.

③ φ is unif. cts on $\mathbb{R}$.

④ $\varphi_{-X}(\theta) = \overline{\varphi_X(\theta)}$.

⑤ $\varphi_{aX+b}(\theta) = e^{ib\theta} \cdot \varphi_X(a\theta)$.

⑥ If $E(|X|^n) < \infty$, then φ is n-th differentiable on $\mathbb{R}$ and

$$\varphi^{(n)}(\theta) = E\left((iX)^n \cdot e^{i\theta X}\right) = \int_{\mathbb{R}} (ix)^n e^{i\theta x} \, d\mu(x).$$

Pf. We only prove ⑥ in the case when $n=1$. First note that

$$|e^{ib}-e^{ia}| \leq b-a \quad \text{for all } a,b\in\mathbb{R},\ a<b. \quad (3)$$

To see it, we have

$$|e^{ib}-e^{ia}| = \left|\frac{1}{i}\int_a^b e^{ix}\,dx\right|$$

$$\leq \int_a^b 1\,dx = b-a.$$

Now for $\varepsilon\neq 0$,

$$\frac{\varphi(\theta+\varepsilon)-\varphi(\theta)}{\varepsilon} = \int_{\mathbb{R}} \frac{e^{i(\theta+\varepsilon)x}-e^{i\theta x}}{\varepsilon}\,d\mu(x)$$

By ③, $\left|\frac{e^{i(\theta+\varepsilon)x}-e^{i\theta x}}{\varepsilon}\right| \leq |x|$

Since $\int |x|\,d\mu(x) = E(|X|)<\infty$, by the DCT,

$$\lim_{\varepsilon\to 0}\frac{\varphi(\theta+\varepsilon)-\varphi(\theta)}{\varepsilon} = \lim_{\varepsilon\to 0}\int_{\mathbb{R}}\frac{e^{i(\theta+\varepsilon)x}-e^{i\theta x}}{\varepsilon}\,d\mu(x)$$

$$= \int_{\mathbb{R}}\left(\lim_{\varepsilon\to 0}\frac{e^{i(\theta+\varepsilon)x}-e^{i\theta x}}{\varepsilon}\right)d\mu(x)$$

$$= \int ix\,e^{i\theta x}\,d\mu(x)$$

$\blacksquare$.

Below we list 3 key results about CFs.

(i) If X and Y are independent, then

$$\varphi_{X+Y} = \varphi_X \cdot \varphi_Y.$$

(ii) F can be reconstructed from φ.

(iii) Weak convergence of DFs corresponds exactly to convergence of the corresponding CFs.

The proof of (i) is straightforward.

The proof of (ii) is based on the following.

Thm 3.8 (Lévy's inversion formula).

Let φ be the CF of a r.v. X with law μ. Then for $a<b$,

$$\lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-i\theta a} - e^{-i\theta b}}{i\theta} \cdot \varphi(\theta)\, d\theta$$

$$= \frac{1}{2}\mu\{a\} + \mu(a,b) + \frac{1}{2}\mu\{b\}.$$

Cor 3.9. Let $\mu, \eta \in \mathrm{Prob}(\mathbb{R})$. Suppose that $\varphi_\mu = \varphi_\eta$. Then $\mu = \eta$.

Pf. By Thm 3.8, for all $a < b$,

$$\tfrac{1}{2}\mu\{a\} + \mu(a,b) + \tfrac{1}{2}\mu\{b\} = \tfrac{1}{2}\eta\{a\} + \eta(a,b) + \tfrac{1}{2}\eta\{b\}.$$

Set

$$\mathcal{A} := \{ A \in \mathcal{B}(\mathbb{R}) : \mu(A) = \eta(A) \}$$

and

$$\mathcal{P} := \{ (a,b] : \mu\{a\} = \mu\{b\} = \eta\{a\} = \eta\{b\},\ \mu(a,b] = \eta(a,b] \}.$$

It is readily checked that $\mathcal{A}$ is a λ-system, and $\mathcal{P}$ is a π-system and $\sigma(\mathcal{P}) = \mathcal{B}(\mathbb{R})$.

By Dynkin's π-λ thm, $\mathcal{A} \supset \sigma(\mathcal{P}) = \mathcal{B}(\mathbb{R})$. Hence $\mu = \eta$.

$\blacksquare$.

Next we turn to the proof of Thm 3.8.

Pf of Thm 3.8:

Let $a<b$ and $T>0$.

$$\frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-i\theta a}-e^{-i\theta b}}{i\theta}\,\varphi(\theta)\,d\theta$$

$$= \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-i\theta a}-e^{-i\theta b}}{i\theta}\cdot\left(\int_{\mathbb{R}} e^{i\theta x}\,d\mu(x)\right)d\theta$$

$$\overset{\text{by Fubini}}{=} \frac{1}{2\pi}\int_{\mathbb{R}}\int_{-T}^{T} \frac{e^{i\theta(x-a)}-e^{i\theta(x-b)}}{i\theta}\,d\theta\,d\mu(x)$$

(Since $\left|\frac{e^{i\theta(x-a)}-e^{i\theta(x-b)}}{i\theta}\right| \leq b-a$,

and $\int_{\mathbb{R}}\int_{-T}^{T} 1\,d\theta\,d\mu(x) < \infty$).

Notice that

$$\int_{-T}^{T}\frac{e^{i\theta y}}{i\theta}\,d\theta = \int_{-T}^{T}\frac{\cos(\theta y)+i\sin(\theta y)}{i\theta}\,d\theta$$

$$= \int_{-T}^{T}\frac{\sin(\theta y)}{\theta}\,d\theta$$

$$= 2\cdot\mathrm{sgn}(y)\cdot\int_{0}^{T|y|}\frac{\sin x}{x}\,dx$$

$$= 2\cdot\mathrm{sgn}(y)\cdot S(T|y|),$$

where $\quad \mathrm{sgn}(y) = \begin{cases} 1 & \text{if } y>0 \\ 0 & \text{if } y=0 \\ -1 & \text{if } y<0 \end{cases}$

and
$$S(z) = \int_0^z \frac{\sin x}{x}\,dx.$$

Notice that
$$\lim_{z\to+\infty} S(z) = \int_0^{+\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2} \qquad (*).$$

Hence
$$\frac{1}{2\pi}\int_{-T}^{T} \frac{e^{i\theta(x-a)} - e^{i\theta(x-b)}}{i\theta}\,d\theta$$

$$= \frac{1}{\pi}\Big[\mathrm{sgn}(x-a)\cdot S\big(T|x-a|\big) - \mathrm{sgn}(x-b)\, S\big(T|x-b|\big)\Big)$$
$$:= A(T,x).$$

$$\xrightarrow{\text{as } T\to\infty} \begin{cases} 1 & \text{if } a<x<b \\ 0 & \text{if } x<a \text{ or } x>b \\ \frac{1}{2} & \text{if } x=a \text{ or } x=b \end{cases}$$

Notice that for any fixed $a<b$, $|A(T,x)|$ is uniformly bdd in T and x.

By the DCT,

$$\lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-i\theta a} - e^{-i\theta b}}{i\theta} \varphi(\theta)\, d\theta$$

$$= \lim_{T\to\infty} \int_{\mathbb{R}} A(T,x)\, d\mu(x)$$

$$\overset{(DCT)}{=} \int_{\mathbb{R}} \frac{1}{2} \mathbb{1}_{\{a\}}(x) + \mathbb{1}_{(a,b)}(x) + \frac{1}{2} \mathbb{1}_{\{b\}}(x)\, d\mu(x)$$

$$= \mu(a,b) + \frac{1}{2}\mu\{a\} + \frac{1}{2}\mu\{b\}.$$

$\blacksquare$

Next we prove (iii), i.e., Weak convergence of DFs correspond to the convergence of the corresponding CFs.

Thm 3.10 (Lévy's convergence Thm)

Let (F_n) be a sequence of DFs, and let φ_n be the CF of F_n.

Suppose that

$$\lim_{n\to\infty} \varphi_n(\theta) = g(\theta) \quad \text{for all} \quad \theta \in \mathbb{R},$$

and that g is cts at 0.

Then $g = \varphi_F$ for some DF F on $\mathbb{R}$ and

$$F_n \xrightarrow{w} F.$$

Pf. We first assume that (F_n) is tight.

By Helly-Bray's Thm, $\exists$ a subsequence (F_{n_k}) and a DF F such that

$$F_{n_k} \xrightarrow{w} F.$$

It follows that

$$\varphi_{n_k}(\theta) \longrightarrow \varphi_F(\theta) \qquad \forall\ \theta \in \mathbb{R},$$

which implies that $g = \varphi_F$.

Next we prove that $F_n \xrightarrow{w} F$. Suppose this is not true.

Then $\exists$ a pt x of continuity of F, $\delta > 0$ and a subsequence (F_{n_k}) such that $|F_{n_k}(x) - F(x)| > \delta$ for all k.

Since (F_n) is tight, choosing a subsequence of (F_{n_k}) if necessary, we may assume that

$$F_{n_k} \xrightarrow{w} \tilde{F}$$

for some DF $\tilde{F}$ on $\mathbb{R}$. Then

$$\varphi_{n_k}(\theta) \to \varphi_{\tilde{F}}(\theta), \quad \forall\, \theta \in \mathbb{R},$$

which implies

$$\varphi_{\tilde{F}} = g = \varphi_F.$$

By Cor 3.9, $\tilde{F} = F$. Hence $F_{n_k} \xrightarrow{w} F$. But

This contradicts the fact that $|F_{n_k}(x) - F(x)| > \delta$ for all k.

In what follows we prove that (F_n) is tight. Notice that $\varphi_n(\theta) + \varphi_n(-\theta)$ is real, so $g(\theta) + g(-\theta)$ is also real.

Let $\varepsilon > 0$. Since g is cts at 0 and $g(0) = 1$, $\exists\ \delta > 0$ such that

$$\frac{1}{\delta} \int_0^\delta 2 - g(\theta) - g(-\theta)\, d\theta < \varepsilon.$$

By the DCT, $\exists\ N \in \mathbb{N}$ such that

$$\frac{1}{\delta} \int_0^\delta 2 - \varphi_n(\theta) - \varphi_n(-\theta)\, d\theta < \varepsilon \quad \text{for all } n \geq N.$$

That is,

$$\frac{1}{\delta} \int_0^\delta \int_{\mathbb{R}} 2 - e^{i\theta x} - e^{-i\theta x}\, d\mu_n(x)\, d\theta < \varepsilon.$$

By Fubini,

$$\int_{\mathbb{R}} \frac{1}{\delta} \int_0^\delta 2 - e^{i\theta x} - e^{-i\theta x}\, d\theta\, d\mu_n(x) < \varepsilon.$$

Notice that $\frac{1}{\delta}\int_0^\delta 2 - e^{i\theta x} - e^{-i\theta x}\, d\theta$

$$= \frac{1}{\delta}\int_0^\delta (2 - 2\cos\theta x)\, d\theta$$

$$= 2 - \frac{1}{\delta}\cdot \frac{2\sin\theta x}{x}\Big|_0^\delta$$

$$= 2\left(1 - \frac{\sin\delta x}{\delta x}\right) \geq 1 \qquad \text{provided that } |\delta x| > 2.$$

Hence

$$\int_{\{x:\ |x| > \frac{2}{\delta}\}} 1 \, d\mu_n \leq \int_{\mathbb{R}} \frac{1}{\delta}\int_0^\delta 2 - e^{i\theta x} - e^{-i\theta x}\, d\theta\, d\mu_n(x) \leq \varepsilon$$

i.e. $\mu_n\left\{x:\ |x| > \frac{2}{\delta}\right\} \leq \varepsilon$ for $n \geq N$.

Equivalently,

$$\mu_n\left(\left[-\frac{2}{\delta}, \frac{2}{\delta}\right]\right) \geq 1 - \varepsilon \quad \text{for all } n \geq N.$$

This proves the tightness of (μ_n). $\boxtimes$