Discrete Fourier Transform:

Definition: The 1D discrete Fourier Transform (DFT) of a function $f(k)$, defined at discrete points $k = 0, 1, 2, \ldots, N-1$ is defined as:

$$\hat{f}(m) = \frac{1}{N}\sum_{k=0}^{N-1} f(k)\, e^{-j\frac{2\pi mk}{N}}$$

(where $j = \sqrt{-1}$, $e^{j\theta} = \cos\theta + j\sin\theta$)

The 2D DFT of a $M \times N$ image $g = (g(k,l))_{k,l}$, where $0 \leq k \leq M-1$, $0 \leq l \leq N-1$ is defined as:

$$\hat{g}(m,n) = \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km}{M} + \frac{ln}{N}\right)}$$

Remark: The inverse of DFT is given by:

$$g(p,q) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \hat{g}(m,n)\, e^{j2\pi\left(\frac{pm}{M} + \frac{qn}{N}\right)}$$

DFT of g

(no $\frac{1}{Mn}$!)

(no -ve sign)

Image decomposition under DFT:

Consider a $N \times N$ image g, the DFT of g:

$$\hat{g}(m,n) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km+ln}{N}\right)}$$

Define $U_{kl} = \frac{1}{N} e^{-j\frac{2\pi kl}{N}}$ where $0 \le k, l \le N-1$ and $U = (U_{kl})_{0 \le k,l \le N-1} \in M_{N \times N}$

U is clearly symmetric and also:

$$\hat{g} = U g U \quad \text{(DFT in matrix form)}$$

Note that: $\frac{1}{N^2} \sum_{\alpha=0}^{N-1} e^{-j\left(\frac{2\pi x_1 \alpha}{N}\right)} e^{+j\left(\frac{2\pi x_2 \alpha}{N}\right)} = \frac{1}{N^2} \sum_{\alpha=0}^{N-1} e^{-j\frac{2\pi (x_2 - x_1)\alpha}{N}}$

$$= \frac{1}{N^2} N \delta(x_2 - x_1)$$

$$= \frac{1}{N} \delta(x_2 - x_1)$$

Let $U = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_N \\ | & | & & | \end{pmatrix}$. Then: $\langle \vec{u}_i, \vec{u}_j \rangle = \vec{u}_i^{\,T} \overline{\vec{u}_j} = \begin{cases} \frac{1}{N} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$\therefore \{\vec{u}_i\}_{i=1}^{N}$ is orthogonal but NOT orthonormal!

Example Find the DFT of the following 4×4 image

$$g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Solution

The matrix U is given by:

$$U = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$$

$$\therefore \text{DFT of } g = \hat{g} = UgU = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U = \left(U_{kl} \right)_{k,l}$$

$$= \frac{1}{4}\left(e^{-j2\pi\left(\frac{kl}{4}\right)} \right)$$

Why is DFT useful in imaging:

1. DFT of convolution:

Recall: $$g * w(n,m) = \sum_{n'=0}^{N-1} \sum_{m'=0}^{N-1} g(n-n', m-m')\, w(n', m')$$

$$\left(g, m \in M_{N \times M}(\mathbb{R}) \right)$$

Then, the DFT of $g * w$ = $MN \, DFT(g) \, DFT(w)$

$\therefore$ DFT of convolution can be reduced to simple multiplication!

Recall: Shift-invariant image transformation = 2D convolution.

$\therefore$ Easy computation/manipulation of shift-invariant transf. after DFT!!

Proof:

DFT of $g*w$ at (p,q)

$$= \frac{1}{NM}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1} g*w(n,m)\, e^{-j2\pi\left(\frac{pn}{N}+\frac{qm}{M}\right)}$$

$$= \frac{1}{NM}\sum_{n'=0}^{N-1}\sum_{m'=0}^{M-1}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1} g(n-n', m-m')\, w(n',m')\, e^{-j2\pi\left(\frac{pn}{N}+\frac{qm}{M}\right)}$$

Change of variables:

$n \rightarrow n'' = n - n'$

$m \rightarrow m'' = m - m'$

$$= \underbrace{\frac{1}{NM}\sum_{n'=0}^{N-1}\sum_{m'=0}^{M-1} w(n',m')\, e^{-j2\pi\left(\frac{pn'}{N}+\frac{qm'}{M}\right)}}_{\hat{w}(p,q)}\ \underbrace{\sum_{n''=-n'}^{N-1-n'}\sum_{m''=-m'}^{M-1-m'} g(n'',m'')\, e^{-j2\pi\left(\frac{pn''}{N}+\frac{qm''}{M}\right)}}_{T(p,q)}$$

Note that: g and w are periodically extended.

$\therefore\ g(n-N, m) = g(n,m)$ and $g(n, m-M) = g(n,m)$

$$\therefore\ T = \sum_{m''=-m'}^{M-1-m'} e^{-j2\pi\frac{qm''}{M}} \sum_{n''=-n'}^{-1} g(n'',m'')\, e^{-j2\pi\frac{pn''}{N}} + \sum_{m''=-m'}^{M-1-m'} e^{-j2\pi\frac{qm''}{M}} \sum_{n''=0}^{N-1-n'} g(n'',m'')\, e^{-j2\pi\left(\frac{pn''}{N}\right)}$$

Consider $\sum_{n''=-n'}^{-1} g(n'',m'')\, e^{-j2\pi\frac{pn''}{N}} \underset{n'''=N+n''}{=} \sum_{n'''=N-n'}^{N-1} \underbrace{g(n'''-N,\, m'')}_{g(n''',\, m'')} e^{-j2\pi\left(\frac{pn'''}{N}\right)} \cancel{e^{j2\pi p}}^{\,1}$

$e^{-j2\pi\left(\frac{pn''}{N}\right)}$, $n'' = n''' - N$

We can do similar thing for index m''.

$$\therefore\ T = \sum_{m''=0}^{M-1} \sum_{n''=0}^{N-1} g(n'',m'')\, e^{-j2\pi\left(\frac{pn''}{N}+\frac{qm''}{M}\right)} = MN\,\hat{g}(p,q)$$

$$\therefore\ \widehat{g*w}(p,q) = MN\,\hat{g}(p,q)\,\hat{w}(p,q)$$

Remark: Conversely, if $x(n,m) = g(n,m)\,w(n,m)$

Then, $\hat{x}(k,l) = \sum_{p=0}^{N-1} \sum_{q=0}^{M-1} \hat{g}(p,q)\,\hat{w}(k-p,\, l-q)$ (Convolution of g and w)

2. Average value of image

Average value of $g = \bar{g} = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} g(k,l) = \underbrace{\frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi(0)}}_{\hat{g}(0,0)}$

3. DFT of a rotated image

Consider a $N \times N$ image g.

Then: $\hat{g}(m,n) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km+ln}{N}\right)}$

Write k and l in polar coordinates:

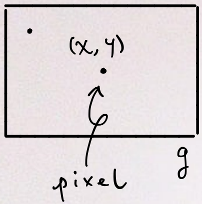
$k \equiv r\cos\theta$; $l \equiv r\sin\theta$

Similarly, write $m \equiv \omega\cos\phi$; $n = \omega\sin\phi$.

Note that: $km + ln = r\omega(\cos\theta\cos\phi + \sin\theta\sin\phi) = r\omega\cos(\theta - \phi)$.

Denote $\mathcal{P}(g) = \{(r,\theta) : (r\cos\theta, r\sin\theta) \text{ is a pixel of } g\}$

(Polar coordinate set of g)



If $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, then $(r, \theta) \in \mathcal{P}(g)$.

Then: $$\underbrace{\hat{g}(m, n) = \hat{g}(\omega, \phi)}_{\text{Identify } \hat{g}(m,n) \text{ with } \hat{g}(\omega,\phi)} = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \underbrace{g(r, \theta)}_{\text{Identify } g(k,l) \text{ with } g(r,\theta)} e^{-j2\pi\left(\frac{r\omega\cos(\theta-\phi)}{N}\right)}$$

Consider a rotated image $\tilde{g}(r, \theta) = g(r, \theta + \theta_0)$ where θ is defined between $-\theta_0$ to $\pi/2 - \theta_0$.

$\therefore$ image g is rotated clockwisely by θ_0.

DFT of $\tilde{g}$ is:

$$\hat{\tilde{g}}(\omega, \phi) = \frac{1}{N^2} \sum_{(r,\theta)\in\mathcal{P}(\tilde{g})} \underset{= g(r,\, \underbrace{\theta+\theta_0}_{\tilde{\theta}})}{\tilde{g}(r, \theta)}\, e^{-j2\pi\left(\frac{r\omega\cos(\theta-\phi)}{N}\right)} = \frac{1}{N^2} \sum_{(r,\tilde{\theta})\in\mathcal{P}(g)} g(r, \tilde{\theta})\, e^{-j2\pi\left(\frac{r\omega\cos(\tilde{\theta}-\theta_0-\phi)}{N}\right)}$$

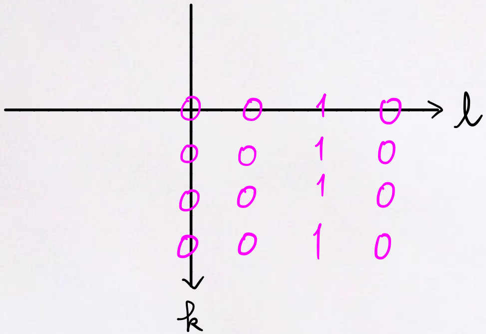
$\therefore \hat{\tilde{g}}(\omega, \phi) = \hat{g}(\omega, \phi + \theta_0)$. (ϕ is also defined between $-\theta_0$ to $\pi/2 - \theta_0$)

DFT of an image rotated by θ_0 = DFT of the original image rotated by θ_0.

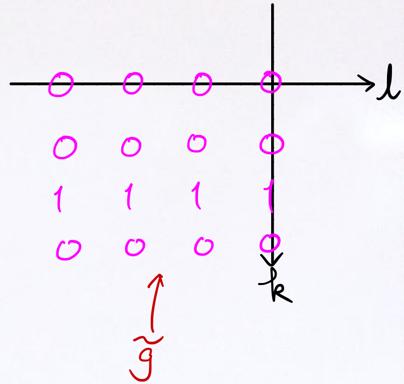
$$0 \le k \le 3$$
$$0 \le l \le 3$$

Example: Let $g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Then: $\hat{g} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Note that g in the coordinate system:



Rotated by 90° clockwisely



Note that indices of $\tilde{g}$ are taken as: $\begin{cases} -3 \le l \le 0 \\ 0 \le k \le 3 \end{cases}$.

Now, DFT of $\tilde{g}$ = $\hat{\tilde{g}}$ (given by: $\displaystyle\sum_{k=0}^{3}\sum_{l=-3}^{0} \tilde{g}(k,l)\, e^{-j2\pi\left(\frac{km+ln}{4}\right)}$

$$= \begin{array}{c c} \begin{pmatrix} 0 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & -1/4 \\ 0 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & -1/4 \end{pmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \ \big\downarrow k \\ l \longrightarrow \begin{array}{cccc} -3 & -2 & -1 & 0 \end{array} & \end{array}$$

$\therefore \quad 0 \le k \le 3$

$-3 \le l \le 0$