Lecture 5:

Recap:

Haar transformation

Definition: (Haar functions) The Haar functions are defined recursively as follows

$$H_0(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

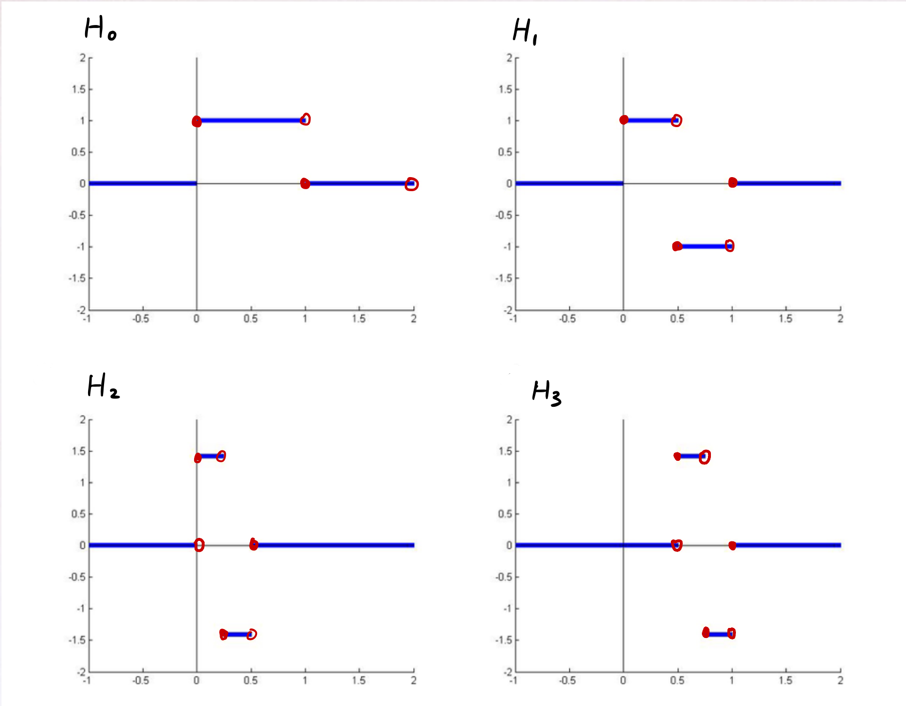
$$H_1(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < 1/2 \\ -1 & \text{if } 1/2 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$H_{2^p+n} \equiv \begin{cases} \sqrt{2}^p & \text{if } \frac{n}{2^p} \le t < \frac{n+0.5}{2^p} \\ -\sqrt{2}^p & \text{if } \frac{n+0.5}{2^p} \le t < \frac{n+1}{2^p} \\ 0 & \text{elsewhere} \end{cases}$$

where $p = 1, 2, \dots;\ n = 0, 1, 2, \dots, 2^p - 1$

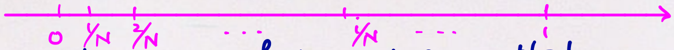
Remark: If p is larger, H_{2^p+n} is compactly supported in a smaller region.

Examples of Haar functions:



Definition (Discrete Haar Transform)

The Haar Transform of a $N \times N$ image is done by dividing $[0,1]$ into partitions.



Let $H(k,i) \equiv H_k\left(\frac{i}{N}\right)$ where $k, i = 0, 1, 2, \ldots, N-1$.

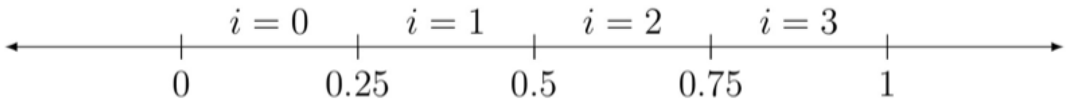
We obtain the Haar Transform matrix: $\tilde{H} \equiv \frac{1}{\sqrt{N}} H$ where $H \equiv \left(H(k,i)\right)_{0 \le k, i \le N-1}$

The Haar Transform of $f \in M_{n \times n}$ is defined as:

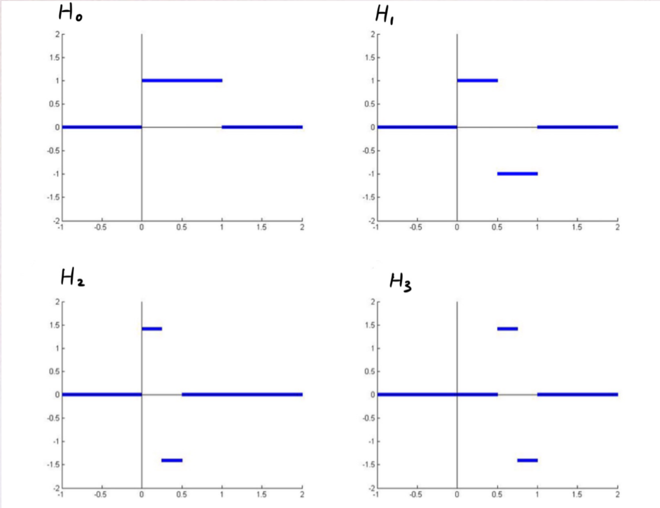
$$g = \tilde{H} f \tilde{H}^T.$$

Example Compute the Haar Transform matrix for a 4×4 image.

Solution: Divide $[0,1]$ into 4 portions:



Need to check:



We get that:

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \quad \text{and} \quad \tilde{H} = \frac{1}{\sqrt{4}} H = \frac{1}{2} H$$

Easy to check that $\tilde{H}^T \tilde{H} = I$.

Elementary images under Haar transform:

Using Haar transform, f can be written as:

$$f = \tilde{H}^T g \tilde{H}$$

transformed image

Let $\tilde{H} = \begin{pmatrix} - & \vec{h}_1^T & - \\ - & \vec{h}_2^T & - \\ & \vdots & \\ - & \vec{h}_N^T & - \end{pmatrix}$. Then: $f = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \vec{h}_i \vec{h}_j^T$

$$\vec{h}_i \vec{h}_j^T = I_{ij}^H$$

I_{ij}^T = elementary images under Haar Transform.

Discrete Fourier Transform:

Definition: The 1D discrete Fourier Transform (DFT) of a function $f(k)$, defined at discrete points $k = 0, 1, 2, \ldots, N-1$ is defined as:

$$\hat{f}(m) = \frac{1}{N} \sum_{k=0}^{N-1} f(k)\, e^{-j\frac{2\pi m k}{N}} \qquad \left(\text{where } j = \sqrt{-1},\ e^{j\theta} = \cos\theta + j\sin\theta\right)$$

The 2D DFT of a $M \times N$ image $g = (g(k,l))_{k,l}$, where $0 \le k \le M-1$, $0 \le l \le N-1$ is defined as:

$$\hat{g}(m,n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km}{M} + \frac{ln}{N}\right)}$$

Remark: The inverse of DFT is given by:

$$g(p,q) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{g}(m,n)\, e^{j2\pi\left(\frac{pm}{M} + \frac{qn}{N}\right)}$$

DFT of g

(no $\frac{1}{Mn}$!)

(no -ve sign)

Image decomposition under DFT:

Consider a $N \times N$ image g, the DFT of g:

$$\hat{g}(m,n) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km+ln}{N}\right)}$$

Define $U_{kl} = \frac{1}{N} e^{-j\frac{2\pi kl}{N}}$ where $0 \le k,l \le N-1$ and $U = (U_{kl})_{0 \le k,l \le N-1} \in M_{N \times N}$

U is clearly symmetric and also:

$$\hat{g} = U g U \quad \text{(DFT in matrix form)}$$

Note that: $\frac{1}{N^2} \sum_{\alpha=0}^{N-1} e^{-j\left(\frac{2\pi x_1 \alpha}{N}\right)} e^{+j\left(\frac{2\pi x_2 \alpha}{N}\right)} = \frac{1}{N^2} \sum_{\alpha=0}^{N-1} e^{-j\frac{2\pi (x_2 - x_1)\alpha}{N}}$

$$= \frac{1}{N^2} N \delta(x_2 - x_1)$$

$$= \frac{1}{N} \delta(x_2 - x_1)$$

Let $U = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_N \\ | & | & & | \end{pmatrix}$. Then: $\langle \vec{u}_i, \vec{u}_j \rangle = \vec{u}_i^{\,T} \overline{\vec{u}_j} = \begin{cases} \frac{1}{N} & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$

$\therefore \{\vec{u}_i\}_{i=1}^{N}$ is orthogonal but NOT orthonormal!

$$\hat{g}(m,n) = \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km+ln}{N}\right)}$$

$$= \underbrace{\sum_{k=0}^{N-1} \underbrace{\frac{e^{-j2\pi\frac{km}{N}}}{N}}_{U_{mk}} \underbrace{\sum_{l=0}^{N-1} g(k,l) \underbrace{\left(\frac{e^{-j\frac{2\pi ln}{N}}}{N}\right)}_{U_{ln}}}_{gU(k,n)}}_{U(gU)(m,n)}$$

$$\therefore \boxed{\hat{g} = UgU}$$

$$\hat{g}(m,n) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km+ln}{N}\right)}$$

$$= \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} g(k,l) \left(\frac{e^{-j2\pi\left(\frac{km}{N}\right)}}{N} \right) \left(\frac{e^{-j2\pi\left(\frac{ln}{N}\right)}}{N} \right)$$

$$= \underbrace{\sum_{k=0}^{N-1} \underbrace{\left(\frac{e^{-j2\pi \frac{km}{N}}}{N} \right)}_{= \, U_{mk}} \underbrace{\left(\sum_{l=0}^{N-1} g(k,l) \underbrace{\left(\frac{e^{-j2\pi \frac{ln}{N}}}{-N} \right)}_{U_{ln}} \right)}_{= \, (gU)(k,n)}}_{= \, U(gU)(m,n)}$$

$$\Longleftrightarrow \quad \hat{g} = UgU$$

$\therefore \quad \mathcal{U}\mathcal{U}^* = \frac{1}{N} I = \mathcal{U}^*\mathcal{U}$

$\therefore \quad g = (N\mathcal{U})^* \, \hat{g} \, (N\mathcal{U})^*$

$\therefore \quad g = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \hat{g}_{kl} \, \vec{w}_k \vec{w}_l^{\,T}$ ← Elementary image of DFT

where $\vec{w}_k = k^{th}$ col of $(N\mathcal{U})^*$

$$\hat{g} = \mathcal{U} \, g \, \mathcal{U}$$

$$\Rightarrow \quad \mathcal{U}^* \hat{g} \, \mathcal{U}^* = (\mathcal{U}^*\mathcal{U}) \, g \, (\mathcal{U}\mathcal{U}^*)$$

$$= \left(\frac{1}{N}\right) g \left(\frac{1}{N}\right)$$

$$\therefore \quad (N\mathcal{U})^* \, \hat{g} \, (N\mathcal{U})^* = g \quad //$$

Example Find the DFT of the following 4×4 image

$$g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Solution

The matrix U is given by:

$$U = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$$

$$U = \left(U_{kl} \right)_{k,l}$$

$$= \frac{1}{4}\left(e^{-j2\pi\left(\frac{kl}{4}\right)} \right)$$

$$\therefore \text{DFT of } g = \hat{g} = UgU = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$