Lecture 2:

Recall:

Let $\mathcal{I}$ = Collection of images of size N and range of intensity $[0, M]$.

$$= \{ f \in M_{N \times N}(\mathbb{R}) : 0 \leq f(i,j) \leq M ; 1 \leq i, j \leq N \}$$

(for simplicity, assume f is a square image; can be general $N_1 \times N_2$ image)

Image transformation = $\mathcal{O} : \mathcal{I} \to \mathcal{I}$ (transform one image to another)

- $\mathcal{O} : \mathcal{I} \to \mathcal{I}$ is linear $\Longleftrightarrow$ $\mathcal{O}(af + g) = a\mathcal{O}(f) + \mathcal{O}(g)$ for $\forall f, g \in \mathcal{I}$; $\forall a \in \mathbb{R}$
- Let $g = \mathcal{O}(f)$. For any $1 \leq \alpha, \beta \leq N$,

$$g(\alpha, \beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} f(x,y)\, h(x, \alpha, y, \beta) \text{ where}$$

$$h(x, \alpha, y, \beta) = [\mathcal{O}(P_{xy})]_{\alpha, \beta} ; \quad P_{xy} = \begin{array}{c} \begin{array}{ccccc} & & y\text{-th} & & \\ & & \downarrow & & \end{array} \\ \left(\begin{array}{ccccc} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{array} \right) \leftarrow x^{th} \end{array}$$

Remark: $h(x, \alpha, y, \beta)$ = how much input value at (x, y) influence the output value at (α, β).

Pixel (x,y) affecting pixel (α, β) by a weight $h(x, \alpha, y, \beta)$.

Definition: (Point spread function)

$h(\cdot, \alpha, \cdot\, \beta)$ is called the PSF at (α, β).

Fix α, β. Let x, y as variables!

Definition: (Shift-invariant)

A PSF is Shift invariant if: (for some $\tilde{h}$ depending on two variables)

$$h(x, \alpha, y, \beta) = \tilde{h}(\alpha - x, \beta - y) \text{ for } \forall\, 1 \le x, y, \alpha, \beta \le N$$

Definition: (Convolution) Let $f, g \in \mathcal{I}$.

Convolution of f and g is defined as $f * g(\alpha, \beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} f(x, y)\, g(\alpha - x, \beta - y)$

(Assume f and g are periodically extended: $\begin{cases} f(x+iN, y+jN) = f(x,y) \\ g(x+iN, y+jN) = g(x,y) \end{cases} \forall\, i, j \in \mathbb{Z}$)

Theorem: If a PSF is shift-invariant, then the operator $\mathcal{O}$ is a convolution with the input image.

Proof: Let $g := \mathcal{O}(f)$. $g(\alpha, \beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} f(x, y) \underbrace{h(x, \alpha, y, \beta)}_{h(\alpha - x,\, \beta - y)}$

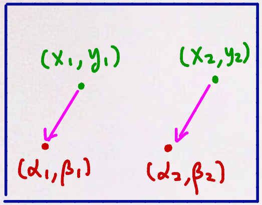
$$= f * h(\alpha, \beta)$$

Remark:

- $f * h = h * f$ (exercise)
- Convolution is important for understanding image blur.

Remark:

- Meaning of shift-invariant PSF $h(x, \alpha, y, \beta)$



Image

Suppose $h(x, \alpha, y, \beta)$ is shift-invariant.

Suppose $(\alpha_1 - x_1, \beta_1 - y_1) = (\alpha_2 - x_2, \beta_2 - y_2)$.

Then:
$$\begin{array}{ccc} h(x_1, \alpha_1, y_1, \beta_1) & = & h(x_2, \alpha_2, y_2, \beta_2) \\ \parallel & & \parallel \\ \tilde{h}(\alpha_1 - x_1, \beta_1 - y_1) & & \tilde{h}(\alpha_2 - x_2, \beta_2 - x_2) \end{array}$$

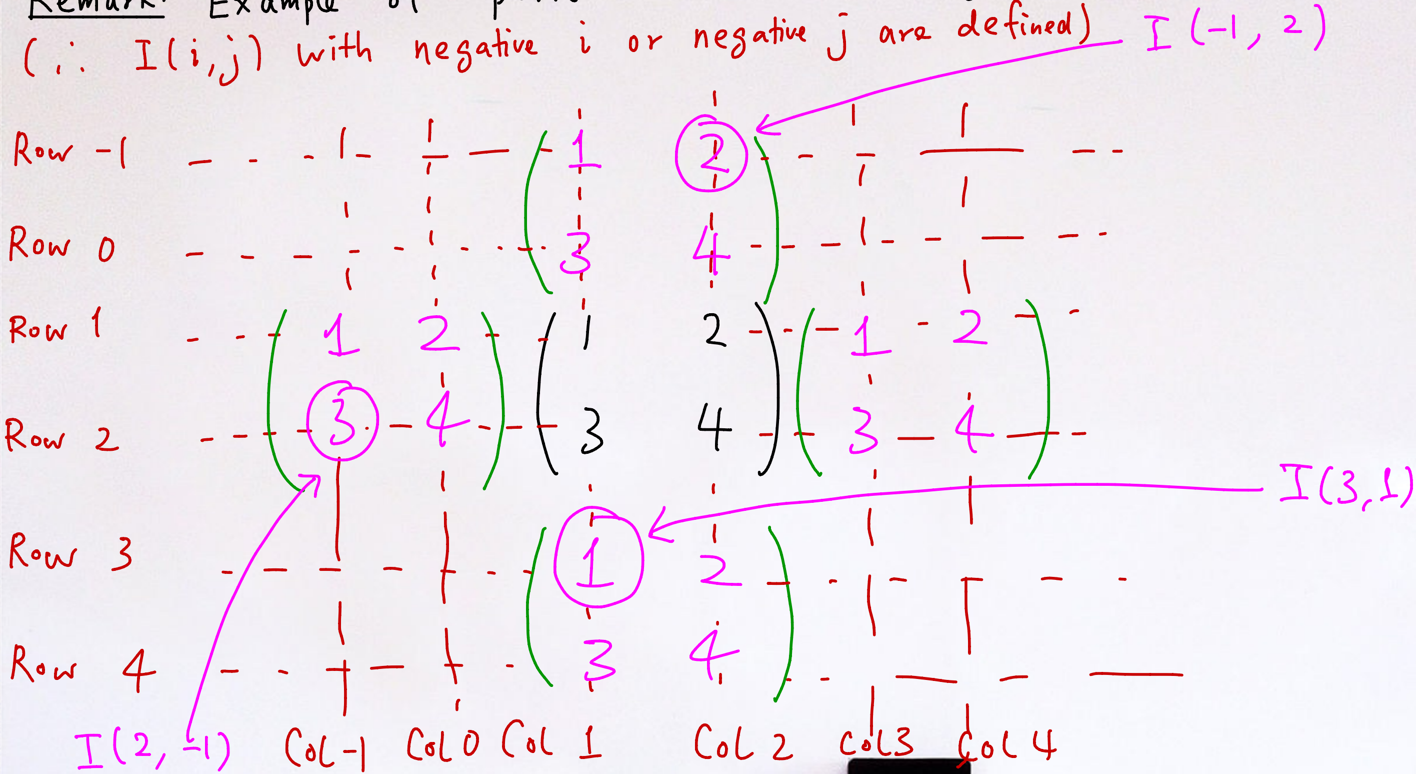
Consider : $g = \mathcal{O}(f)$.

Intuitively, it means the influence of $f(x_1, y_1)$ on $g(\alpha_1, \beta_1)$ is the same as the influence of $f(x_2, y_2)$ on $g(\alpha_2, \beta_2)$!

In other words, PSF $h(x, \alpha, y, \beta)$ depends only on the displacement between (α, β) and (x, y) (That is, $(\alpha - x, \beta - y)$)

Remark: Example of periodic extension of image I

($\because$ $I(i,j)$ with negative i or negative j are defined)



More about convolution

"Geometric" intrepretation of discrete convolution

Let $f * g(\alpha, \beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} f(x,y)\, g(\alpha - x, \beta - y)$

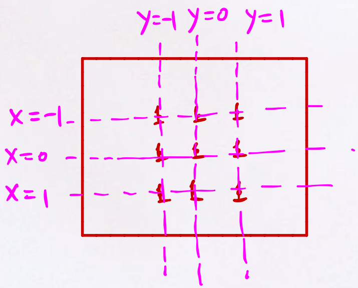
Consider a simple case where only several entries of g are non-zero.

Namely,

$g(0,0) = g(N,N) = 1$; $g(1,0) = g(1,N) = 1$; $g(-1,0) = g(N-1,N) = 1$

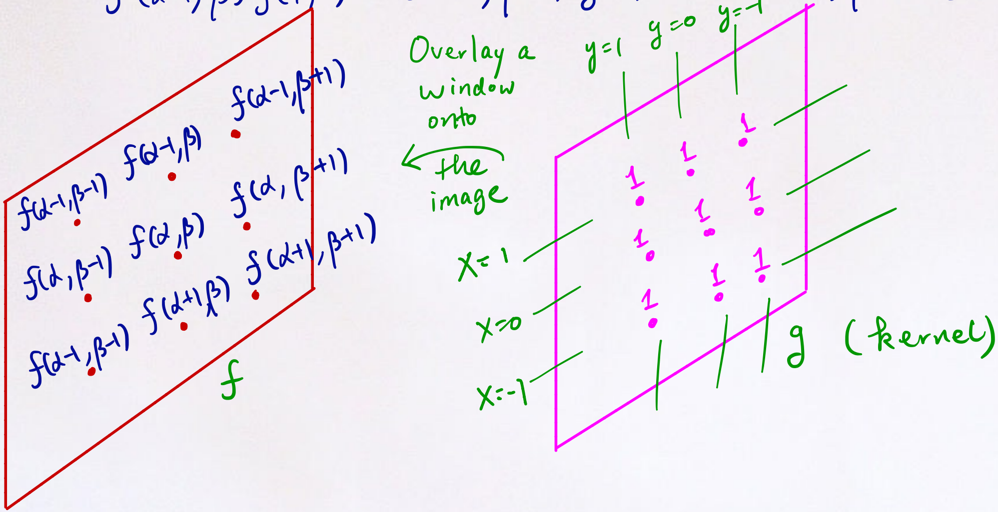
$g(0,1) = g(N,1) = 1$; $g(1,1) = g(1,1) = 1$; $g(-1,1) = g(N-1,1) = 1$

$g(0,-1) = g(N,N-1) = 1$; $g(1,-1) = g(1,N-1) = 1$; $g(-1,-1) = g(N-1,N-1) = 1$



Expand the summation:

$$
\begin{aligned}
f*g(\alpha,\beta) = {} & f(\alpha,\beta)\,g(0,0) + f(\alpha,\beta+1)\,g(0,-1) + f(\alpha,\beta-1)\,g(0,1) + \\
& f(\alpha+1,\beta)\,g(-1,0) + f(\alpha+1,\beta+1)\,g(-1,-1) + f(\alpha+1,\beta-1)\,g(-1,1) + \\
& f(\alpha-1,\beta)\,g(1,0) + f(\alpha-1,\beta+1)\,g(1,-1) + f(\alpha-1,\beta-1)\,g(1,1).
\end{aligned}
$$



Representation of $\mathcal{O}$ by a matrix H :

We can write :

$$
\left.
\begin{aligned}
g(\alpha,\beta) = {} & f(1,1)\,h(1,\alpha,1,\beta) + f(2,1)\,h(2,\alpha,1,\beta) + \dots + f(N,1)\,h(N,\alpha,1,\beta) \\
& + f(1,2)\,h(1,\alpha,2,\beta) + \dots + f(N,2)\,h(N,\alpha,2,\beta) \\
& \dots \\
& + f(1,N)\,h(1,\alpha,N,\beta) + \dots + f(N,N)\,h(N,\alpha,N,\beta)
\end{aligned}
\right\}
$$

Each (α,β) is associated to a linear equations.

Arrange:

$$
\vec{f} = \begin{pmatrix} f(1,1) \\ \vdots \\ f(N,1) \\ f(1,2) \\ \vdots \\ f(N,2) \\ \vdots \\ f(1,N) \\ \vdots \\ f(N,N) \end{pmatrix} ; \quad
\vec{g} = \begin{pmatrix} g(1,1) \\ g(2,1) \\ \vdots \\ g(N,1) \\ g(1,2) \\ \vdots \\ g(N,2) \\ \vdots \\ g(1,N) \\ g(2,N) \\ \vdots \\ g(N,N) \end{pmatrix} .
$$

In matrix form, let

$$
\vec{g} = \begin{pmatrix} g(1,1) \\ \vdots \\ g(N,1) \\ \vdots \\ g(1,N) \\ \vdots \\ g(N,N) \end{pmatrix} .
$$

Then : $\vec{g} = H \vec{f}$

$N^2 \times N^2$ matrix

Example 1.1 A linear operator is such that it replaces the value of each pixel by the average of its four nearest neighbours. Assume the image is repeated in all directions. Apply this operator $\mathcal{O}$ to a 3×3 image. Find the transformation matrix corresponding to $\mathcal{O}$.

Solution:

$$
3 \times 3 \text{ image} = \begin{array}{ccccc} & f_{31} & f_{32} & f_{33} & \\ f_{13} & \multicolumn{3}{c}{\multirow{3}{*}{$\begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix}$}} & f_{11} \\ f_{23} & & & & f_{21} \\ f_{33} & & & & f_{31} \\ & f_{11} & f_{12} & f_{13} & \end{array}
$$

$$
g_{22} = \frac{f_{12} + f_{21} + f_{23} + f_{32}}{4} \; ; \quad g_{33} = \frac{f_{23} + f_{32} + f_{31} + f_{13}}{4}
$$

etc ...

By careful examination, we see that

$$\underbrace{\begin{pmatrix} g_{11} \\ g_{21} \\ g_{31} \\ g_{12} \\ g_{22} \\ g_{32} \\ g_{13} \\ g_{23} \\ g_{33} \end{pmatrix}}_{\vec{g}} = \begin{bmatrix} 0 & 1/4 & 1/4 & 1/4 & 0 & 0 & 1/4 & 0 & 0 \\ 1/4 & 0 & 1/4 & 0 & 1/4 & 0 & 0 & 1/4 & 0 \\ 1/4 & 1/4 & 0 & 0 & 0 & 1/4 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 0 & 0 \\ 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 1/4 & 0 & 0 & 0 & 1/4 & 1/4 \\ 0 & 1/4 & 0 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 \\ 0 & 0 & 1/4 & 0 & 0 & 1/4 & 1/4 & 1/4 & 0 \end{bmatrix} \underbrace{\begin{pmatrix} f_{11} \\ f_{21} \\ f_{31} \\ f_{12} \\ f_{22} \\ f_{32} \\ f_{13} \\ f_{23} \\ f_{13} \end{pmatrix}}_{\vec{f}}$$

By careful examination, we see that:

$$
\begin{array}{cc}
 & \xrightarrow{\hspace{6em} y \hspace{6em}} \\
 & \begin{array}{cccc} \in M_{N\times N} & \in M_{N\times N} & & \in M_{N\times N} \end{array} \\
\beta \Big\downarrow & H = \begin{pmatrix}
\begin{pmatrix} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=1 \\ \beta=1 \end{pmatrix} \end{pmatrix} & \begin{pmatrix} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=2 \\ \beta=1 \end{pmatrix} \end{pmatrix} & \cdots & \begin{pmatrix} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=N \\ \beta=1 \end{pmatrix} \end{pmatrix} \\
\begin{pmatrix} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=1 \\ \beta=2 \end{pmatrix} \end{pmatrix} & \begin{pmatrix} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=2 \\ \beta=2 \end{pmatrix} \end{pmatrix} & \cdots & \begin{pmatrix} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=N \\ \beta=2 \end{pmatrix} \end{pmatrix} \\
\vdots & \vdots & & \vdots \\
\begin{pmatrix} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=1 \\ \beta=N \end{pmatrix} \end{pmatrix} & \begin{pmatrix} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=2 \\ \beta=N \end{pmatrix} \end{pmatrix} & \cdots & \begin{pmatrix} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=N \\ \beta=N \end{pmatrix} \end{pmatrix}
\end{pmatrix} \in M_{N^2\times N^2}
\end{array}
$$

Meaning of

$$
\begin{array}{cccc}
\text{col of small block} & \text{row of small block} & \text{col of block matrix} & \text{row of block matrix} \\
\downarrow & \downarrow & \swarrow & \swarrow \\
\end{array}
$$

$$h(x, \alpha, y, \beta)$$

$$
\begin{pmatrix} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=i \\ \beta=j \end{pmatrix} \end{pmatrix} :
\begin{pmatrix}
h(1,1,i,j) & h(2,1,i,j) & \cdots & h(N,1,i,j) \\
h(1,2,i,j) & h(2,2,i,j) & \cdots & h(N,2,i,j) \\
\vdots & \vdots & & \vdots \\
h(1,N,i,j) & h(2,N,i,j) & \cdots & h(N,N,i,j)
\end{pmatrix}
$$

$$\in M_{N\times N}$$

<u>Definition</u>: H is called the <u>transformation matrix</u> of $\mathcal{O}$.

By careful examination, we see that

$$\begin{bmatrix} 0 & 1/4 & 1/4 & 1/4 & 0 & 0 & 1/4 & 0 & 0 \\ 1/4 & 0 & 1/4 & 0 & 1/4 & 0 & 0 & 1/4 & 0 \\ 1/4 & 1/4 & 0 & 0 & 0 & 1/4 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 0 & 0 \\ 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 1/4 & 0 & 0 & 0 & 1/4 & 1/4 \\ 0 & 1/4 & 0 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 \\ 0 & 0 & 1/4 & 0 & 0 & 1/4 & 1/4 & 1/4 & 0 \end{bmatrix}$$

What is $h(2,3,2,1)$?

What is $h(1,2,3,3)$?

$h(2,3,2,1) = 0$

$h(1,2,3,3) = 1/4.$

Recall:

$$H = \begin{pmatrix} \left(\alpha\downarrow \overset{x\rightarrow}{\begin{pmatrix} y=1 \\ \beta=1 \end{pmatrix}}\right) & \left(\alpha\downarrow \overset{x\rightarrow}{\begin{pmatrix} y=2 \\ \beta=1 \end{pmatrix}}\right) & \cdots & \left(\alpha\downarrow \overset{x\rightarrow}{\begin{pmatrix} y=N \\ \beta=1 \end{pmatrix}}\right) \\ \left(\alpha\downarrow \overset{x\rightarrow}{\begin{pmatrix} y=1 \\ \beta=2 \end{pmatrix}}\right) & \left(\alpha\downarrow \overset{x\rightarrow}{\begin{pmatrix} y=2 \\ \beta=2 \end{pmatrix}}\right) & \cdots & \left(\alpha\downarrow \overset{x\rightarrow}{\begin{pmatrix} y=N \\ \beta=2 \end{pmatrix}}\right) \\ \vdots & \vdots & & \vdots \\ \left(\alpha\downarrow \overset{x\rightarrow}{\begin{pmatrix} y=1 \\ \beta=N \end{pmatrix}}\right) & \left(\alpha\downarrow \overset{x\rightarrow}{\begin{pmatrix} y=2 \\ \beta=N \end{pmatrix}}\right) & \cdots & \left(\alpha\downarrow \overset{x\rightarrow}{\begin{pmatrix} y=N \\ \beta=N \end{pmatrix}}\right) \end{pmatrix}$$

Definition: (Separable)

A PSF is Separable if: $h(x,\alpha,y,\beta) = h_c(x,\alpha)\, h_r(y,\beta)$ for $\forall \quad 1 \le x, y, \alpha, \beta \le N$.

Theorem: Suppose PSF is separable. Then:

The operator $\mathcal{O}$ consists of two matrix multiplication.

Proof:

$$g(\alpha,\beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} f(x,y)\, h(x,\alpha,y,\beta)$$

$$= \sum_{x=1}^{N} \sum_{y=1}^{N} f(x,y)\, h_c(x,\alpha)\, h_r(y,\beta)$$

$$= \underbrace{\sum_{x=1}^{N} h_c(x,\alpha) \Big(\underbrace{\sum_{y=1}^{N} f(x,y)\, h_r(y,\beta)}_{\text{matrix multiplication}} \Big)}_{\text{matrix multiplication}}$$

$$\boxed{\text{Recall: } (AB)_{ij} = \sum_{k=1}^{N} A_{ik} B_{kj}}$$

Properties of separable image transformation

Recall: Separable $h \Leftrightarrow h(x,\alpha,y,\beta) = h_c(x,\alpha)\, h_r(y,\beta)$.

Let $g = Hf$.

H: transformation matrix

$$\Longrightarrow g(\alpha,\beta) = \sum_{x=1}^{N} h_c(x,\alpha) \underbrace{\sum_{y=1}^{N} f(x,y)\, h_r(y,\beta)}_{\text{Matrix multiplication}}$$

Consider $h_r = (h_r(y,\beta))_{1\le y,\beta\le N} \in M_{N\times N}$

$h_c = (h_c(x,\alpha))_{1\le x,\alpha\le N} \in M_{N\times N}$ Let $S = f\, h_r$.

$f = (f(x,y))_{1\le x,y\le N} \in M_{N\times N}$

Easy to see: $g(\alpha,\beta) = \sum_{x=1}^{N} h_c(x,\alpha)\, S(x,\beta) = \sum_{x=1}^{N} h_c^T(\alpha,x)\, S(x,\beta)$

$\therefore\ g = h_c^T S = h_c^T f\, h_r$ (Matrix form)

Image decomposition

Suppose $h(x,\alpha,y,\beta) = h_c(x,\alpha)\, h_r(y,\beta)$ (Separable).

Then: $g = h_c^T f h_r \Rightarrow f = (h_c^T)^{-1} g (h_r)^{-1}$

Write: $(h_c^T)^{-1} = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_N \\ | & | & & | \end{pmatrix}$; $h_r^{-1} = \begin{pmatrix} - & \vec{v}_1^T & - \\ - & \vec{v}_2^T & - \\ & \vdots & \\ - & \vec{v}_N^T & - \end{pmatrix}$

Then: $f = \underbrace{\sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij}\, \vec{u}_i \vec{v}_j^T}$ $M_{N\times N}$

Check that: $(h_c^T)^{-1} \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 0 \end{pmatrix} h_r^{-1} = \vec{u}_i \vec{v}_j^T$

(i,j)-entry

$\therefore f =$ linear combination of $\{\vec{u}_i \vec{v}_j^T\}_{i,j}$