Lecture 1:

Image transformation

Let $\mathcal{I}$ = Collection of images of size N and range of intensity $[0, M]$.

$$= \{ f \in M_{N\times N}(\mathbb{R}) : 0 \leq f(i,j) \leq M ; 1 \leq i, j \leq N \}$$

(for simplicity, assume f is a square image; can be general $N_1 \times N_2$ image)

Image transformation = $\mathcal{O} : \mathcal{I} \to \mathcal{I}$ (transform one image to another)

Imaging problems

① Find a suitable transformation $T \ni$ $g := T(f)$ becomes good!

noisy image

② Given a noisy/distorted image g and transformation T, find original clean image f.

$$g = T(f) + n$$

known, known, unknown, unknown

(Inverse problem)

Definition: (Linear image transformation)

$\mathcal{O}:\mathcal{I}\to\mathcal{I}$ is linear $\iff$ $\mathcal{O}(af+g) = a\,\mathcal{O}(f)+\mathcal{O}(g)$ for $\forall f,g\in\mathcal{I}$; $\forall a\in\mathbb{R}$

Take $f\in\mathcal{I}$. Let

$$f = \begin{pmatrix} f(1,1) & \cdots & f(1,N) \\ f(2,1) & & f(2,N) \\ \vdots & f(i,j) & \vdots \\ f(N,1) & \cdots & f(N,N) \end{pmatrix} = \sum_{i=1}^{N}\sum_{j=1}^{N}\begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & f(i,j) & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} = \sum_{i=1}^{N}\sum_{j=1}^{N} f(x,y)\begin{array}{c} \downarrow y \\ \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \end{array}\begin{array}{c} \\ \leftarrow x \end{array}$$

Let $g=\mathcal{O}(f)$. Assume $\mathcal{O}$ is linear, then:

$$g(\alpha,\beta) = \sum_{x=1}^{N}\sum_{y=1}^{N} f(x,y)\left[\mathcal{O}\left(\left(\begin{matrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{matrix}\right)\right)\right]_{\alpha,\beta}$$

$$= \sum_{x=1}^{N}\sum_{y=1}^{N} f(x,y)\, h(x,\alpha,y,\beta) \quad \text{where}$$

$$h(x,\alpha,y,\beta) = \left[\mathcal{O}(P_{xy})\right]_{\alpha,\beta} \; ; \; P_{xy} = \begin{array}{c} y^{th} \\ \downarrow \\ \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \end{array}\begin{array}{c} \\ \\ \leftarrow x^{th} \end{array}$$

Remark: $h(x,\alpha,y,\beta)$ = how much input value at (x,y) influence the output value at (α,β).

Pixel (x,y) affecting pixel (α,β) by a weight $h(x,\alpha,y,\beta)$.

Definition: (Point spread function)

$h(\cdot,\alpha,\cdot\,\beta)$ is called the PSF at (α,β).

Fix α,β. Let x,y as variables!

Definition: (Shift-invariant)

A PSF is Shift invariant if:

$$h(x,\alpha,y,\beta) = h(\alpha-x,\ \beta-y) \text{ for } \forall\, 1 \le x,y,\alpha,\beta \le N$$

Definition: (Convolution) Let $f, g \in \mathcal{I}$.

Convolution of f and g is defined as $f * g(\alpha, \beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} f(x,y)\, g(\alpha - x, \beta - y)$

(Assume f and g are periodically extended: $\begin{cases} f(x+iN, y+jN) = f(x,y) \\ g(x+iN, y+jN) = g(x,y) \end{cases} \forall\, i,j \in \mathbb{Z}$)

Theorem: If a PSF is shift-invariant, then the operator $\mathcal{O}$ is a convolution with the input image.

Proof: Let $g := \mathcal{O}(f)$.

$$g(\alpha, \beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} f(x,y)\, \underbrace{h(x, \alpha, y, \beta)}_{h(\alpha - x,\ \beta - y)}$$

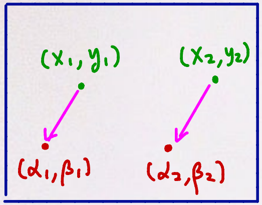
$$= f * h(\alpha, \beta)$$

Remark:

- $f * h = h * f$ (exercise)
- Convolution is important for understanding image blur.

Remark:

- Meaning of shift-invariant PSF $h(x, \alpha, y, \beta)$



Image

Suppose $h(x,\alpha,y,\beta)$ is shift-invariant.

Suppose $(\alpha_1 - x_1, \beta_1 - y_1) = (\alpha_2 - x_2, \beta_2 - y_2)$.

Then:
$$\begin{array}{ccc} h(x_1,\alpha_1,y_1,\beta_1) & = & h(x_2,\alpha_2,y_2,\beta_2) \\ \| & & \| \\ \tilde{h}(\alpha_1 - x_1, \beta_1 - y_1) & & \tilde{h}(\alpha_2 - x_2, \beta_2 - x_2) \end{array}$$

Consider : $g = \mathcal{O}(f)$.

Intuitively, it means the influence of $f(x_1, y_1)$ on $g(\alpha_1, \beta_1)$ is the same as the influence of $f(x_2, y_2)$ on $g(\alpha_2, \beta_2)$!

In other words, PSF $h(x, \alpha, y, \beta)$ depends only on the displacement between (α, β) and (x, y) (That is, $(\alpha - x, \beta - y)$)

Remark: Example of periodic extension of image I

($\because$ I(i,j) with negative i or negative j are defined)