MMAT5390: Mathematical Image Processing

Assignment 1

Due: 13 February 2023

Please give reasons in your solutions.

1. (a) For the following transformation matrices, determine if their corresponding point spread functions (PSF) are separable and/or shift-invariant (with periodical extension assumption). If not, give your reason.

   i. $H = \begin{pmatrix} 2 & 0 & 5 & 3 \\ 2 & 3 & 9 & 0 \\ 5 & 3 & 2 & 0 \\ 9 & 0 & 2 & 3 \end{pmatrix}$;

   ii. $H = \begin{pmatrix} 1 & 5 & 2 & 2 & 10 & 4 & 0 & 0 & 0 \\ 6 & 8 & 9 & 12 & 16 & 18 & 0 & 0 & 0 \\ 7 & 0 & 4 & 14 & 0 & 8 & 0 & 0 & 0 \\ 4 & 20 & 8 & 3 & 15 & 6 & 2 & 10 & 4 \\ 24 & 32 & 36 & 18 & 24 & 27 & 12 & 16 & 18 \\ 28 & 0 & 16 & 21 & 0 & 12 & 14 & 0 & 8 \\ 1 & 5 & 2 & 0 & 0 & 0 & 2 & 10 & 4 \\ 6 & 8 & 9 & 0 & 0 & 0 & 12 & 16 & 18 \\ 7 & 0 & 4 & 0 & 0 & 0 & 14 & 0 & 8 \end{pmatrix}$;

   (b) Consider the following point spread functions, determine whether they are separable and/or shift-invariant. If not, provide your reason.

   i. $h(x, \alpha, y, \beta) = \frac{\alpha}{\beta} ln(x^y)$;

   ii. $h(x, \alpha, y, \beta) = ln(\alpha - x) - \frac{1}{\beta - y}$

2. (a) Suppose the transformation matrix $H = \begin{pmatrix} 1 & 2 & 2 & 4 \\ 0 & 7 & 0 & 14 \\ 4 & 8 & 1 & 2 \\ 0 & 28 & 0 & 7 \end{pmatrix}$, is this transformation separable? If yes, find out g_1 and g_2 such that $h(x, \alpha, y, \beta) = g_1(x, \alpha) g_2(y, \beta)$;

   (b) Consider the general case, if the PSF of a linear image transformation for $N \times N$ images is given by $h(x, \alpha, y, \beta) = h_c(x, \alpha) h_r(y, \beta)$, show that the transformation matrix $H = h_r^T \otimes h_c^T$.

   Definition: Suppose A and B are two matrices. The **Kronecker product** $A \otimes B$ is defined as:

   $$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1N}B \\ a_{21}B & \cdots & a_{2N}B \\ \vdots & & \vdots \\ a_{N1}B & \cdots & a_{NN}B \end{pmatrix},$$

   where a_{ij} is the i-th row, j-th column entry of A.

3. Let $f = \begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix}$ and $g = \begin{pmatrix} 6 & 5 \\ 4 & 2 \end{pmatrix}$, assume f and g are periodically extended, $*$ denotes convolution operation.

   (a) Compute $f * g$ and $g * f$.

   (b) Show that $f * g = g * f$ when f and g are two $m \times n$ images with periodical extension.

4. (a) Let H be the transformation matrix of shift-invariant transformation on $N \times N$ images. Assume that the images are periodically extended. Show that H is *block-circulant.*

   (b) Here we consider a general case of (a). Let H be the transformation matrix of shift-invariant transformation on $N \times N$ images. We do not assume that the images are periodically extended. Show that H is *block-Toeplitz.*

   Remark: a Toeplitz matrix is a matrix in which each descending diagonal from left to right is constant. For example, the following matrix T is Toeplitz

$$T = \begin{pmatrix} a & b & c & d \\ e & a & b & c \\ f & e & a & b \\ g & f & e & a \end{pmatrix}$$

   A block Toeplitz matrix is another special kind of block matrix, which contains blocks that are repeated down the diagonals of the matrix, as a Toeplitz matrix has elements repeated down the diagonal. The individual block matrix elements, A_{ij}, must also be a Toeplitz matrix. A block-Toeplitz matrix has the form

$$A = \begin{pmatrix} A_0 & A_{-1} & \cdots & \\ A_1 & A_0 & A_{-1} & \cdots \\ A_2 & A_1 & A_0 & \cdots \\ \cdots & \cdots & & \end{pmatrix}, \text{ where matrices } A_0, A_1, \cdots \text{ are Toeplitz.}$$

5. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

   (a) Compute an SVD of A. Please show all your steps.

   (b) Write A as a linear combination of its elementary images from SVD. Please show all your steps.