MMAT5390: Mathematical Image Processing

Assignment 2 Solutions

1. (a) $\{\sigma_{11}, \sigma_{22}, \ldots, \sigma_{KK}\}$ are the square roots of the eigenvalues of A^TA or AA^T, whichever is smaller. As the entries of the K-tuple are listed in descending order, the K-tuple is uniquely determined.

   More detailed version:

   Recall that an SVD of $A \in M_{M\times N}(\mathbb{R})$ involves a triple $(U, \Sigma, V) \in O(M) \times M_{M\times N}(\mathbb{R}) \times O(N)$ such that $A = U\Sigma V^T$, and Σ is a diagonal matrix (in the sense that $\sigma_{ij} = 0$ whenever $i \neq j$) with all entries nonnegative.

   Let $A \in M_{M\times N}(\mathbb{R})$, and let (U_1, Σ_1, V_1) and (U_2, Σ_2, V_2) represent two SVDs of A, i.e.

$$A = U_1\Sigma_1V_1^T = U_2\Sigma_2V_2^T,$$

   which satisfy

$$(\Sigma_i)_{kk} \geq (\Sigma_i)_{k+1,k+1}, \quad i = 1, 2, \quad k = 1, 2, \cdots, K-1.$$

   Then

$$U_1\Sigma_1\Sigma_1^TU_1^T = U_1\Sigma_1V_1^TV_1\Sigma_1^TU_1^T = AA^T = U_2\Sigma_2V_2^TV_2\Sigma_2^TU_2^T = U_2\Sigma_2\Sigma_2^TU_2^T,$$

   and

$$V_1\Sigma_1^T\Sigma_1V_1^T = V_1\Sigma_1^TU_1^TU_1\Sigma_1V_1^T = A^TA = V_2\Sigma_2^TU_2^TU_2\Sigma_2V_2^T = V_2\Sigma_2^T\Sigma_2V_2^T.$$

   The multiset of eigenvalues of a square matrix is uniquely determined by the characteristic polynomial of the matrix, which in turn only depends on the matrix.

   Since both $\Sigma_1\Sigma_1^T$ and $\Sigma_2\Sigma_2^T$ are diagonal matrices similar to AA^T, the multiset of diagonal entries of either of the matrices is the multiset of eigenvalues of AA^T, which implies $\Sigma_1\Sigma_1^T = \Sigma_2\Sigma_2^T$ due to their ordering. Similarly, $\Sigma_1^T\Sigma_1 = \Sigma_2^T\Sigma_2$. The result follows from considering the square roots of the diagonal entries of $\begin{cases} \Sigma_1\Sigma_1^T & \text{if } M \leq N \\ \Sigma_1^T\Sigma_1 & \text{if } M \geq N \end{cases}$.

   (b) Suppose $\{\sigma_{ii} : i = 1, 2, \cdots, K\}$ are distinct *and nonzero*. Then each eigenspace of A^TA and AA^T corresponding to eigenvalue σ_{ii}^2 has dimension 1, which means that there are exactly two unit eigenvectors to be chosen from each eigenspace, each being the negative of the other. Such eigenvectors are precisely the first K columns of U and V. Combined with the fact that σ_{ii} are in descending order, the first K columns of U and V are uniquely determined up to a change of sign.

   (Apologies, the claim would be incorrect without the assumption that σ_{KK} is nonzero. When A is not a square matrix, the eigenspace of A^TA or AA^T, whichever has more rows and columns, that correspond to the eigenvalue 0 would have dimension greater than 1. Hence there would be more than two choices of unit eigenvectors from this eigenspace. One example would be

$$(0\ 0) = (1)(0\ 0)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (1)(0\ 0)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.)$$

   (c) A simple counterexample is given by:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2I_2I_2 = UI_2U^T,$$

   where I_2 and $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ are unitary.

2. (a) $H_0(t) = \mathbf{1}_{[0,1)}$, and for any $p \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{Z} \cap [0, 2^p - 1]$,

$$H_{2^p+n}(t) = 2^{\frac{p}{2}} \left(\mathbf{1}_{[\frac{n}{2^p}, \frac{n+0.5}{2^p})} - \mathbf{1}_{[\frac{n+0.5}{2^p}, \frac{n+1}{2^p})} \right).$$

(b) $\tilde{H} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}$.

(c)

$$\begin{aligned}
A_{\text{Haar}} &= \tilde{H} A \tilde{H}^T \\
&= \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 2 \\ 2 & 3 & 3 & 4 \\ 2 & 3 & 3 & 4 \\ 4 & 5 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} 8 & 12 & 12 & 16 \\ -4 & -4 & -4 & -4 \\ -2\sqrt{2} & -2\sqrt{2} & -2\sqrt{2} & -2\sqrt{2} \\ -2\sqrt{2} & -2\sqrt{2} & -2\sqrt{2} & -2\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} 12 & -2 & -\sqrt{2} & -\sqrt{2} \\ -4 & 0 & 0 & 0 \\ -2\sqrt{2} & 0 & 0 & 0 \\ -2\sqrt{2} & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

Then the modified Haar transform A'_{Haar} is $\begin{pmatrix} 0 & -2 & -\sqrt{2} & -\sqrt{2} \\ -4 & 0 & 0 & 0 \\ -2\sqrt{2} & 0 & 0 & 0 \\ -2\sqrt{2} & 0 & 0 & 0 \end{pmatrix}$, and thus:

$$\begin{aligned}
\tilde{A} &= \tilde{H}^T A'_{\text{Haar}} \tilde{H} \\
&= \frac{1}{4} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & -2 & -\sqrt{2} & -\sqrt{2} \\ -4 & 0 & 0 & 0 \\ -2\sqrt{2} & 0 & 0 & 0 \\ -2\sqrt{2} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} -8 & -2 & -\sqrt{2} & -\sqrt{2} \\ 0 & -2 & -\sqrt{2} & -\sqrt{2} \\ 0 & -2 & -\sqrt{2} & -\sqrt{2} \\ 8 & -2 & -\sqrt{2} & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} -3 & -2 & -2 & -1 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 3 \end{pmatrix}.
\end{aligned}$$

3. (a) The 2D discrete Fourier transform (DFT) of an $M \times N$ image $g = (g(k,l))_{k,l}$, where $k = 0, 1, \cdots, M-1$ and $l = 0, 1, \cdots, N-1$, is defined as:

$$\hat{g}(m,n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g(k,l) e^{-2\pi j (\frac{km}{M} + \frac{ln}{N})}$$

The Fourier transform matrix $U = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$.

(b)

$$
\begin{aligned}
C_{\text{DFT}} &= UCU \\
&= \frac{1}{16}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}\begin{pmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \\
&= \begin{pmatrix} 7.5 & -\frac{1}{2}+\frac{1}{2}j & -\frac{1}{2} & -\frac{1}{2}-\frac{1}{2}j \\ -2+2j & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -2-2j & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

(c) The modified Fourier coefficient matrix

$$
\tilde{C}_{\text{DFT}} = \begin{pmatrix} 7.5 & -\frac{1}{2}+\frac{1}{2}j & 0 & -\frac{1}{2}-\frac{1}{2}j \\ -2+2j & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -2-2j & 0 & 0 & 0 \end{pmatrix}
$$

Thus,

$$
\begin{aligned}
\tilde{C} &= (4U^*)\tilde{C}_{\text{DFT}}(4U^*) \\
&= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix}\begin{pmatrix} 7.5 & -\frac{1}{2}+\frac{1}{2}j & 0 & -\frac{1}{2}-\frac{1}{2}j \\ -2+2j & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -2-2j & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix} \\
&= \begin{pmatrix} 0.5 & 0.5 & 2.5 & 2.5 \\ 4.5 & 4.5 & 6.5 & 6.5 \\ 8.5 & 8.5 & 10.5 & 10.5 \\ 12.5 & 12.5 & 14.5 & 14.5 \end{pmatrix}
\end{aligned}
$$

4. (a) So the DFT transformed g is

$$
\widehat{g} = UgU = \frac{1}{16}\begin{pmatrix} 8 & 5-3j & 2 & 5+3j \\ 8 & 5-3j & 2 & 5+3j \\ 8 & 5-3j & 2 & 5+3j \\ 8 & 5-3j & 2 & 5+3j \end{pmatrix}
$$

(b) Note that $\widehat{f * g} = 16\widehat{f} \otimes \widehat{g}$, where $\otimes$ denotes entrywise matrix multiplication. Thus, it is obvious that

$$
\widehat{f} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

Then, we have

$$
\begin{aligned}
f &= (4U^*)\widehat{f}(4U^*) \\
&= \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}
\end{aligned}
$$

5. **Coding Assignment:**

MATLAB:

```
recon = (h * U') * freq * (h * U');
```

Python:

```
recon = (h * U.T.conjugate()) @ freq @ (h * U.T.conjugate())
```