MMAT5390: Mathematical Image Processing

Assignment 1 solutions

1. (a) i. Note that H is a 4×4 matrix; hence it represents a linear transformation on 2×2 images.
H is not block-circulant. For example, consider the $y=1,\beta=1$-submatrix of H, i.e. $\begin{pmatrix}2 & 0\\ 2 & 3\end{pmatrix}$. This is not a circulant matrix, as the shift-operator T maps $\begin{pmatrix}2\\2\end{pmatrix}$ to $\begin{pmatrix}0\\3\end{pmatrix}$ instead of $\begin{pmatrix}2\\2\end{pmatrix}$. Hence h is not shift-invariant with h_s being 2-periodic in both arguments.

H is not a Kronecker product of two 2×2 matrices. For example, consider the $y=1,\beta=1$- and $y=2,\beta=1$-submatrices of H, i.e. $\begin{pmatrix}2 & 0\\ 2 & 3\end{pmatrix}$ and $\begin{pmatrix}5 & 3\\ 9 & 0\end{pmatrix}$. Neither is a scalar multiple of the other. Hence h is not separable.

ii. Note that H is a 9×9 matrix; hence it represents a linear transformation on 3×3 images.
H is not block-circulant. For example, consider the $y=1,\beta=1$-submatrix of h, i.e. $\begin{pmatrix}1 & 5 & 2\\ 6 & 8 & 9\\ 7 & 0 & 4\end{pmatrix}$, which is not a circulant matrix, as the shift-operator t maps $\begin{pmatrix}1\\6\\7\end{pmatrix}$ to $\begin{pmatrix}7\\1\\6\end{pmatrix}$ instead of $\begin{pmatrix}5\\8\\0\end{pmatrix}$. Hence h is not shift-invariant with h_s being 3-periodic in both arguments. (Neither is H block-toeplitz, hence neither is h shift-invariant.)
H is the kronecker product of two 3×3 matrices; explicitly,

$$H = \begin{pmatrix}1 & 2 & 0\\ 4 & 3 & 2\\ 1 & 0 & 2\end{pmatrix} \otimes \begin{pmatrix}1 & 5 & 2\\ 6 & 8 & 9\\ 7 & 0 & 4\end{pmatrix}$$

Hence h is separable.

(b) i. Note that $h(1,2,1,1)=0$, $h(2,3,1,1)=3\ln 2$, $h(1,2,1,1)\neq h(2,3,1,1)$, hence H is not shift-invariant.
Let $g_1(x,\alpha)=\alpha\ln x$ and $g_2(y,\beta)=\frac{y}{\beta}$, then $h(x,\alpha,y,\beta)=g_1(x,\alpha)g_2(y,\beta)$, hence it is separable.

ii. Let $s=\alpha-x, t=\beta-y$. Then, $h(x,\alpha,y,\beta)=ln(s)-\frac{1}{t}$. Hence, h is shift-invariant.
Suppose h is separable. Then, there exists h_c, h_r such that $h(x,\alpha,y,\beta)=h_c(x,\alpha)h_r(y,\beta)$.
We can then deduce the following results:
$h(1,2,2,3)=h_c(1,2)h_r(2,3)=-1$ and $h(1,2,2,4)=h_c(1,2)h_r(2,4)=-\frac{1}{2}$
$\implies h_r(2,3)/h_r(2,4)=2$.
But, $h(1,3,2,3)=h_c(1,3)h_r(2,3)=\ln 2-1$ and $h(1,3,2,4)=h_c(1,3)h_r(2,4)=\ln 2-\frac{1}{2}$
$\implies h_r(2,3)/h_r(2,4)=\frac{\ln 2-1}{\ln 2-\frac{1}{2}}\neq 2$.
Hence, h is not separable.

2. (a) Note that
$h(1,1,1,1)=1$, $h(2,1,1,1)=2$, $h(1,1,2,1)=2$, $h(2,1,2,1)=4$,
$h(1,2,1,1)=0$, $h(2,2,1,1)=7$, $h(1,2,2,1)=0$, $h(2,2,2,1)=14$,
$h(1,1,1,2)=4$, $h(2,1,1,2)=8$, $h(1,1,2,2)=1$, $h(2,1,2,2)=2$,
$h(1,2,1,2)=0$, $h(2,2,1,2)=28$, $h(1,2,2,2)=0$, $h(2,2,2,2)=7$.
Define $g_1:\{1,2\}^2\to\mathbb{R}$ by $g_1(1,1)=1$, $g_1(2,1)=2$, $g_1(1,2)=0$, $g_1(2,2)=7$.

and $g_2 : \{1,2\}^2 \to \mathbb{R}$ by $g_2(1,1) = 1,\ g_2(2,1) = 2,\ g_2(1,2) = 4,\ g_2(2,2) = 1$
As $h(x,\alpha,y,\beta) = g_1(x,\alpha)g_2(y,\beta)$ for all $1 \leq x,\alpha,y,\beta) \leq 2$, h is separable.
And we observe that

$$H = \begin{pmatrix} 1 & 2 & 2 & 4 \\ 0 & 7 & 0 & 14 \\ 4 & 8 & 1 & 2 \\ 0 & 28 & 0 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 2 \\ 0 & 7 \end{pmatrix}.$$

(b) Let h be the separable PSF of a linear image transformation, with $h(x,\alpha,y,\beta) = h_c(x,\alpha)h_r(y,\beta)$. Let H be the corresponding transformation matrix.
Then the $y = k, \beta = l$-submatrix of H (denoted by $\tilde{H}_{kl}$) is given by

$$\left(\begin{matrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = k \\ \beta = l \end{pmatrix} \end{matrix} \right) = [H\,(\alpha + (l-1)n, x + (k-1)n)]_{\substack{1 \leq x \leq n \\ 1 \leq \alpha \leq n}}$$
$$= [h(x,\alpha,k,l)]_{\substack{1 \leq x \leq n \\ 1 \leq \alpha \leq n}}$$
$$= [h_c(x,\alpha)h_r(k,l)]_{\substack{1 \leq x \leq n \\ 1 \leq \alpha \leq n}}$$
$$= h_r(k,l)[h_c(x,\alpha)]_{\substack{1 \leq x \leq n \\ 1 \leq \alpha \leq n}}$$
$$= h_r(k,l)h_c^T.$$

Recall that

$$H = \begin{pmatrix} \left(\begin{matrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = 1 \\ \beta = 1 \end{pmatrix} \end{matrix} \right) & \left(\begin{matrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = 2 \\ \beta = 1 \end{pmatrix} \end{matrix} \right) & \cdots & \left(\begin{matrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = n \\ \beta = 1 \end{pmatrix} \end{matrix} \right) \\ \left(\begin{matrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = 1 \\ \beta = 2 \end{pmatrix} \end{matrix} \right) & \left(\begin{matrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = 2 \\ \beta = 2 \end{pmatrix} \end{matrix} \right) & \cdots & \left(\begin{matrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = n \\ \beta = 2 \end{pmatrix} \end{matrix} \right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(\begin{matrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = 1 \\ \beta = n \end{pmatrix} \end{matrix} \right) & \left(\begin{matrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = 2 \\ \beta = n \end{pmatrix} \end{matrix} \right) & \cdots & \left(\begin{matrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = n \\ \beta = n \end{pmatrix} \end{matrix} \right) \end{pmatrix}$$
$$= \begin{pmatrix} \tilde{H}_{11} & \tilde{H}_{21} & \cdots & \tilde{H}_{n1} \\ \tilde{H}_{12} & \tilde{H}_{22} & \cdots & \tilde{H}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{H}_{1n} & \tilde{H}_{2n} & \cdots & \tilde{H}_{nn} \end{pmatrix} = \begin{pmatrix} h_r(1,1)h_c^T & h_r(2,1)h_c^T & \cdots & h_r(n,1)h_c^T \\ h_r(1,2)h_c^T & h_r(2,2)h_c^T & \cdots & h_r(n,2)h_c^T \\ \vdots & \vdots & \ddots & \vdots \\ h_r(1,n)h_c^T & h_r(2,n)h_c^T & \cdots & h_r(n,n)h_c^T \end{pmatrix}$$

$$= \begin{pmatrix} h_r^T(1,1)h_c^T & h_r^T(1,2)h_c^T & \cdots & h_r^T(1,n)h_c^T \\ h_r^T(2,1)h_c^T & h_r^T(2,2)h_c^T & \cdots & h_r^T(2,n)h_c^T \\ \vdots & \vdots & \ddots & \vdots \\ h_r^T(n,1)h_c^T & h_r^T(n,2)h_c^T & \cdots & h_r^T(n,n)h_c^T \end{pmatrix} = h_r^T \otimes h_c^T.$$

3. (a) For simplicity (and to guide the indexing of $f * g$), we only consider the cases where f and g are indexed with the same set of indices.
If f and g are indexed with $1 \leq i,j \leq 2$, i.e. if

$$f = (f(i,j))_{1 \leq i,j \leq 2} = \begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix} \text{ and } g = (g(i,j))_{1 \leq i,j \leq 2} = \begin{pmatrix} 6 & 5 \\ 4 & 2 \end{pmatrix},$$

then

$$\begin{aligned}
f*g(1,1) &= f(1,1)g(2,2) + f(1,2)g(2,1) + f(2,1)g(1,2) + f(2,2)g(1,1)\\
&= 1\cdot 2 + 0\cdot 4 + 2\cdot 5 + 4\cdot 6 = 36,\\
f*g(1,2) &= f(1,1)g(2,1) + f(1,2)g(2,2) + f(2,1)g(1,1) + f(2,2)g(1,2)\\
&= 1\cdot 4 + 0\cdot 2 + 2\cdot 6 + 4\cdot 5 = 36,\\
f*g(2,1) &= f(1,1)g(1,2) + f(1,2)g(1,1) + f(2,1)g(2,2) + f(2,2)g(2,1)\\
&= 1\cdot 5 + 0\cdot 6 + 2\cdot 2 + 4\cdot 4 = 25, \text{ and}\\
f*g(2,2) &= f(1,1)g(1,1) + f(1,2)g(1,2) + f(2,1)g(2,1) + f(2,2)g(2,2)\\
&= 1\cdot 6 + 0\cdot 5 + 2\cdot 4 + 4\cdot 2 = 22,
\end{aligned}$$

i.e. $f*g = (f*g(i,j))_{1\le i,j\le 2} = \begin{pmatrix} 36 & 36\\ 25 & 22\end{pmatrix}$.

$$\begin{aligned}
g*f(1,1) &= g(1,1)f(2,2) + g(1,2)f(2,1) + g(2,1)f(1,2) + g(2,2)f(1,1)\\
&= 6\cdot 4 + 5\cdot 2 + 4\cdot 0 + 2\cdot 1 = 36,\\
g*f(1,2) &= g(1,1)f(2,1) + g(1,2)f(2,2) + g(2,1)f(1,1) + g(2,2)f(1,2)\\
&= 6\cdot 2 + 5\cdot 4 + 4\cdot 1 + 2\cdot 0 = 36,\\
g*f(2,1) &= g(1,1)f(1,2) + g(1,2)f(1,1) + g(2,1)f(2,2) + g(2,2)f(2,1)\\
&= 6\cdot 0 + 5\cdot 1 + 4\cdot 4 + 2\cdot 2 = 25, \text{ and}\\
g*f(2,2) &= g(1,1)f(1,1) + g(1,2)f(1,2) + g(2,1)f(2,1) + g(2,2)f(2,2)\\
&= 6\cdot 1 + 5\cdot 0 + 4\cdot 2 + 2\cdot 4 = 22,
\end{aligned}$$

i.e. $g*f = (g*f(i,j))_{1\le i,j\le 2} = \begin{pmatrix} 36 & 36\\ 25 & 22\end{pmatrix}$.

(b) Let $f, g \in M_{m\times n}(\mathbb{R})$, and assume that they are periodically extended.
Let $\alpha \in \mathbb{N}\cap[1,m]$ and $\beta \in \mathbb{N}\cap[1,n]$. By definition,

$$\begin{aligned}
f*g(\alpha,\beta) &= \sum_{x=1}^{m}\sum_{y=1}^{n} f(x,y)g(\alpha-x,\beta-y)\\
&= \sum_{i=\alpha-m}^{\alpha-1}\sum_{j=\beta-n}^{\beta-1} f(\alpha-i,\beta-j)g(i,j) \text{ (letting } i=\alpha-x,\ j=\beta-y)\\
&= \sum_{i=\alpha-m}^{0}\sum_{j=\beta-n}^{0} f(\alpha-i,\beta-j)g(i,j) + \sum_{i=\alpha-m}^{0}\sum_{j=1}^{\beta-1} f(\alpha-i,\beta-j)g(i,j)\\
&\quad + \sum_{i=1}^{\alpha-1}\sum_{j=\beta-n}^{0} f(\alpha-i,\beta-j)g(i,j) + \sum_{i=1}^{\alpha-1}\sum_{j=1}^{\beta-1} f(\alpha-i,\beta-j)g(i,j)\\
&= \sum_{i=\alpha}^{m}\sum_{j=\beta}^{n} f(\alpha-i,\beta-j)g(i,j) + \sum_{i=\alpha}^{m}\sum_{j=1}^{\beta-1} f(\alpha-i,\beta-j)g(i,j)\\
&\quad + \sum_{i=1}^{\alpha-1}\sum_{j=\beta}^{n} f(\alpha-i,\beta-j)g(i,j) + \sum_{i=1}^{\alpha-1}\sum_{j=1}^{\beta-1} f(\alpha-i,\beta-j)g(i,j) \text{ (by periodicity)}\\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} g(i,j)f(\alpha-i,\beta-j)\\
&= g*f(\alpha,\beta);
\end{aligned}$$

hence $f*g = g*f$.

4. (a) Let h be the shift-invariant PSF of a linear image transformation on $M_{n\times n}(\mathbb{R})$, with h_s n-periodic in both arguments such that $h(x,\alpha,y,\beta) = h_s(\alpha-x,\beta-y)$. Let H be the corresponding transformation matrix.

Let $a \in \mathbb{Z}$. Then

$$\begin{aligned} H\left(\alpha + (\beta + a - 1)n, x + (y + a - 1)n\right) &= h(x, \alpha, y + a, \beta + a) \\ &= h_s\left(\alpha - x, (\beta + a) - (y + a)\right) \\ &= h_s(\alpha - x, \beta - y) \\ &= h(x, \alpha, y, \beta) \\ &= H\left(\alpha + (\beta - 1)n, x + (y - 1)n\right). \end{aligned}$$

Also, by periodicity of h_s, for $y \in \mathbb{N} \cap [1, n-1]$,

$$\begin{aligned} H(\alpha + (n-1)n, x + (y-1)n) &= h(x, \alpha, y, n) \\ &= h_s(\alpha - x, n - y) \\ &= h_s(\alpha - x, 1 - (y + 1) \\ &= h(x, \alpha, y + 1, 1) \\ &= H(\alpha, x + yn) \end{aligned}$$

and for $\beta \in \mathbb{N} \cap [1, n-1]$,

$$\begin{aligned} H(\alpha + (\beta - 1)n, x + (n-1)n) &= h(x, \alpha, n, \beta) \\ &= h_s(\alpha - x, \beta - n) \\ &= h_s(\alpha - x, (\beta + 1) - 1) \\ &= h(x, \alpha, 1, \beta + 1) \\ &= H(\alpha + \beta n, x). \end{aligned}$$

Hence H is circulant when viewed as a matrix consisting of blocks of fixed (y, β)-values. Combined with the result of Theorem 1.13, we establish that H is block-circulant.

(b) Let h be the shift-invariant PSF of a linear image transformation on $M_{n\times n}(\mathbb{R})$ in the sense that $h(x, \alpha, y, \beta) = h_s(\alpha - x, \beta - y)$. Let H be the corresponding transformation matrix.
Fix y and β. Then for any α, x and $a \in \mathbb{N}$ satisfying $a \leq n - \max\{\alpha, x\}$,

$$\begin{aligned} h\left(x + an, \alpha + an, y, \beta\right) &= h_s\left(\alpha + an - x - an, \beta - y\right) \\ &= h_s(\alpha - x, \beta - y) \\ &= h(x, \alpha, y, \beta) \end{aligned}$$

On the other hand, fix x and α. Then for any β, y and $a \in \mathbb{N}$ satisfying $a \leq n - \max\{\beta, y\}$,

$$\begin{aligned} h(\alpha, x, \beta + an, y + an) &= h_s(\alpha - x, \beta + an - y - an) \\ &= h_s(\alpha - x, \beta - y) \\ &= h(x, \alpha, y, \beta) \end{aligned}$$

Hence, we know H is block Toeplitz.
Reverse all the statements shown above, we know h is shift-invariant if H is block Toeplitz.

5. (a) $AA^T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has $\sigma_1^2 = 3$ with $u_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ and $\sigma_2^2 = 1$ with $u_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$.
$A^TA = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ has $\sigma_1^2 = 3$ with $v_1 = \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$, $\sigma_2^2 = 1$ with $v_2 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$ and $v_3 = \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$.
Then $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}^T$

(b) We can then write A as $A = \sqrt{3}u_1v_1^T + u_2v_2^T + 0u_3v_3^T = \sqrt{3}\begin{pmatrix} 1/\sqrt{12} & 2/\sqrt{12} & 1/\sqrt{12} \\ 1/\sqrt{12} & 2/\sqrt{12} & 1/\sqrt{12} \end{pmatrix} + \begin{pmatrix} 1/2 & 0 & -1/2 \\ -1/2 & 0 & 1/2 \end{pmatrix}$