MMAT5390: Mathematical Image Processing

Assignment 2

Due: March 2 2023

Please give reasons in your solutions.

1. Consider a real $M \times N$ matrix A, and denote one of its singular value decompositions as

$$A = U\Sigma V^T$$

such that $\sigma_{ii} \geq \sigma_{jj}$ whenever $i < j$.

   (a) Show that the K-tuple $(\sigma_{11}, \sigma_{22}, \ldots, \sigma_{KK})$, where $K = \min\{M, N\}$, is uniquely determined.

   (b) Show that if all the singular values are distinct and nonzero, then the *first* K columns of U and V are uniquely determined up to a change of sign. In other words, for each $i = 1, 2, \ldots, K$, there are exactly two choices of $(\vec{u}_i, \vec{v}_i)$; denoting one choice by $(\vec{u}, \vec{v})$, the other is given by $(-\vec{u}, -\vec{v})$.

   (c) Does the claim in (b) hold if we drop the assumption? Prove it or give a counterexample.

2. Let $H_n(t)$ be the n^{th} Haar function, where $n \in \mathbb{N} \cup \{0\}$.

   (a) Write down the definition of $H_n(t)$.

   (b) Write down the Haar transform matrix $\tilde{H}$ for 4×4 images.

   (c) Suppose $A = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 2 & 3 & 3 & 4 \\ 2 & 3 & 3 & 4 \\ 4 & 5 & 5 & 6 \end{pmatrix}$. Compute the Haar transform A_{Haar} of A, and compute the reconstructed image $\tilde{A}$ after setting the largest entry of A_{Haar} to 0.

3. (a) Write down the definition of 2D DFT of an $M \times N$ image, and give the Fourier transform matrix U for 4×4 images.

   (b) Let $C = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{pmatrix}$. Using the matrix U derived above to compute the Fourier transform C_{DFT} of C.

   (c) By setting the smallest (in modulus value) nonzero elements of C_{DFT} to 0, we obtain $\tilde{C}_{\text{DFT}}$. Compute the reconstructed image $\tilde{C}$ of $\tilde{C}_{\text{DFT}}$.

4. (a) Using the Fourier transform matrix U obtained in last question, compute the DFT of the following image:

$$g = \begin{pmatrix} 5 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   (b) Suppose there is another image $f \in \mathbb{R}^{4\times 4}$ such that

$$\widehat{f * g} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   Find f.

5. **Coding assignment (Optional)**: Please read the MATLAB file **or** the Jupyter notebook file in the attached zip file carefully. There are missing lines in the file. You can either choose MATLAB or Python to finish. Add the missing lines by yourself and test the file using the given image. (Note: In this coding assignment, we discuss the image processing of grayscale images only.)

   Coding instruction:

   Q1: Recall that DFT can be rewritten as matrix multiplication.

   $$\hat{g} = UgU \tag{1}$$

   where $U_{\alpha\beta} = \frac{1}{N} e^{-2\pi j \frac{\alpha\beta}{N}}$ where $0 \leq \alpha, \beta \leq N-1$, and $U = (U_{\alpha\beta})_{0 \leq \alpha,\beta \leq N-1} \in M_{N \times N}(\mathbb{C})$.

   In this coding assignment, you are required to reconstruct the image given a modified $\hat{g}$, which represents the Fourier coefficients. You are **not allowed** to use the built-in MATLAB function $ifft2$ or any Python Fourier transform module such as numpy.fft module and scipy.fft.

   Later in this course, we will do image processing in the spectral domain. We will use this technique again and again.