MATH5070
Homework 1 solution

1. (i) We use the notation in the following questions. Identify M with $\mathcal{M}$. Let $f_I : \mathbb{R}^{k\times n} \to \mathbb{R}$, $A \mapsto det(A_I)$, which is continuous, then $\mathcal{M}_I = f_I^{-1}(\mathbb{R} \setminus \{0\})$ is an open set since $\mathbb{R} \setminus \{0\}$ is an open set. Therefore, $\mathcal{M} = \cup_I \mathcal{M}_I$ is an open set of a kn-dimensional vector space.

   (ii) By definition.

2. $(i) \implies (ii)$ A_1 is E-equivalent to A_2 means image(A_1)=image(A_2), i.e., the column space of A_1 can be represented by linear combinations of column vectors of A_2. Therefore there exists matrix B such that $A_1 = A_2B$. B has to be invertible since Rank(A_1)=Rank(A_2).
   $(ii) \implies (iii)$ By the fact from linear algebra that any invertible matrix can be written in terms of a multiplication of a sequence of elementary matrices.
   $(iii) \implies (i)$ Multiplying elementary matrices on the right doesn't change the column space of a matrix.

3. (i) $\mathcal{M}_I = \mathcal{M} \cap \mathcal{M}_I$ and $\mathcal{M}_{\mathcal{I}}$ is an open set of $\mathbb{R}^{k\times n}$ as in Q1, thus $\mathcal{M}_I$ is an open set of $\mathcal{M}$.

   (ii) By the fact that any $n \times k$ matrix A is of rank k if and only if that there exists a $k \times k$ minor of A is invertible.

   (iii) By Q2, any matrix which is E-equivalent to A is of the form AB where B is an invertible $k \times k$ matrix. Also $(AB)_I = A_IB$, and A_IB is invertible, therefore $AB \in \mathcal{M}_I$.

4. $A^\# \triangleq AA_I^{-1}$, and the uniqueness is due to the fact that inverse of a matrix is unique.

5. The map $\psi_I : W_I \to \mathcal{U}_I$, $A \mapsto [A]$ is bijective due to the existence and uniqueness of $A^\#$. The identification $\phi_I : W_I \cong \mathbb{R}^{k(n-k)}$ is defined by omitting the identity minor matrix in W_I. Then $\varphi_I \triangleq \phi_I \circ \psi_I^{-1}$.

6. The transition map $\varphi_{I_2} \circ \varphi_{I_1}^{-1} = \phi_{I_2} \circ \psi_{I_2}^{-1} \circ \phi_{I_1} \circ \phi_{I_1}^{-1}$ is basically a rational function, and rational function is smooth. Note that a rational function is any function which can be defined by a rational fraction such that both the numerator and the denominator are polynomials.