Homework 4 for MATH5070
Topology of Manifolds
Due Wednesday, Nov. 2

1. A vector bundle $E \to M$ is a *trivial bundle* if it admits a global trivialization, i.e., there is a global smooth diffeomorphism $\Phi : E \to M \times \mathbb{R}^k$ so that for each p, $\Phi|_{E_p} : E_p \to \mathbb{R}^k \times \{p\}$ is a linear isomorphism. Prove:

   (i) A vector bundle E is trivial if and only if it admits a global frame, i.e., k smooth sections $s_1, \cdots, s_k$ over M so that for each $p \in M$, $s_1(p), \cdots, s_k(p)$ form a basis of E_p.

   (ii) For any Lie group G, the tangent bundle TG is a trivial bundle.

2. An *inner product* on a vector bundle $E \to M$ is a map which assigns to each $p \in M$ an inner product $\langle\, , \rangle_p$ on the fiber E_p. This inner product is *smooth* if for every pair of smooth sections s_1 and s_2, the function $p \to \langle s_1(p), s_2(p)\rangle_p$ is smooth. Show that every vector bundle can be equipped with a smooth inner product.

   Hint: Let $\{U_\alpha, \alpha \in I\}$ be an open covering of M, and for each $\alpha \in I$ let $\{s_1^\alpha, \cdots, s_k^\alpha\}$ be a trivialization of E_{U_α}. Then there is a unique inner product $\langle\, , \rangle_\alpha$ on E_{U_α} for which $s_1^\alpha(p), \cdots, s_k^\alpha(p)$ is an orthonormal basis of E_p for all $p \in U_\alpha$. Let $\{\rho_\alpha, \alpha \in I\}$ be a partition of unity subordinate to the above covering. Show that the sum

   $$\sum \rho_\alpha \langle\, , \rangle_\alpha$$

   makes sense and defines an inner product on E.

3. If M is compact, show that every vector bundle $E \to M$ is a sub-bundle of the trivial bundle $M \times \mathbb{R}^N \to M$ for some large integer N.

   Hint: Show that the dual bundle $E^* \to M$ admits a set of global section $s_1, \cdots, s_N$, $N \geq \operatorname{rank} E$ such that for every $p \in M$, the vectors $s_1(p), \cdots, s_N(p)$ span E_p^*.

4. Let M be a compact manifold and $E \to M$ a vector bundle over M. Show that that exists a vector bundle $F \to M$ having the property that the direct sum $E \oplus F$ is the trivial bundle $M \times \mathbb{R}^N \to M$.

5. ("Pull-backs" of vector bundles) Let M and N be manifolds and $f : M \to N$ a smooth map. Given a smooth vector bundle $E \to N$, let

   $$f^*E_p = E_{f(p)}$$

   for every point $p \in M$. Show that the vector bundle $f^*E \to M$ with fibers above is a smooth vector bundle.

6. ("Canonical bundle") Let $M_k(\mathbb{R}^n)$ be the Grassmannian of k-dimensional subspace of $\mathbb{R}^n$. Recall that a point p of $M_k(\mathbb{R}^n)$ is by definition a k-dimensional vector subspace E_p of $\mathbb{R}^n$. (To avoid confusing "points of $M_k(\mathbb{R}^n)$" with "k-dimensional subspaces of $\mathbb{R}^n$", we will use p, q, etc. for the former and E_p, E_q, etc. for the latter.) Let

$$E \to M_k(\mathbb{R}^n)$$

be the rank k vector bundle whose fiber at p is E_p. Prove that this is a smooth vector bundle. This bundle is called the *canonical bundle*.

Hint: HW#1, exercise 5.

7. Denote by E_{can} the canonical bundle on $M_k(\mathbb{R}^n)$. Let M be a compact manifold and $E \to M$ a rank k vector bundle. Show that there exists an integer N and a smooth map

$$f : M \to M_k(\mathbb{R}^N)$$

such that

$$E = f^* E_{can}.$$

Hint: Exercise 3.

8. (Optional) Suppose G is a Lie group with Lie algebra $\mathfrak{g}$. Define their centers by

$$Z(G) = \{z \in G \mid gz = zg \text{ for all } g \in G\}$$

and

$$Z(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X, Y] = 0 \text{ for all } Y \in \mathfrak{g}\}.$$

Show that

(i) $Z(G)$ is a Lie subgroup of G.

(ii) $Z(\mathfrak{g})$ is a Lie subalgebra of $\mathfrak{g}$.

(iii) The Lie algebra of $Z(G)$ is $Z(\mathfrak{g})$.

9. (Optional) Let $G = SL(2, \mathbb{R})$. Prove:

(i) As smooth manifolds, G is diffeomorphic to $S^1 \times \mathbb{R}^2$.

(ii) As Lie groups, G is not isomorphic to $S^1 \times \mathbb{R}^2$ (as a product Lie group).

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