# Homework 3 for MATH5070

# Topology of Manifolds

**Due Wednesday, Oct. 19**

1. (i) Let $A$ be an $n \times n$ matrix. Show that the infinite series

$$\exp tA = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

converges uniformly on compact subintervals of the $t$-axis.

(ii) Show that $\exp tA$ is differentiable as a function of $t$ and that

$$\frac{d}{dt}\exp tA = (\exp tA)A = A(\exp tA).$$

*Hint*: First show that if one differentiates the series above term by term, one gets a series which is uniformly convergent on compact intervals.

(iii) Conclude from (ii) that $\exp tA$ is smooth in $t$.

2. Let $A = (a_{ij})$ be an $n \times n$ matrix and let $v_A$ be the vector field on $\mathbb{R}^n$:

$$v_A = \sum (a_{ij}x_j)\frac{\partial}{\partial x_i}$$

Show that $v_A$ generates a global one-parameter group of diffeomorphisms of $\mathbb{R}^n$.
*Hint*: Let $x_0$ be an arbitrary point of $\mathbb{R}^n$. Show that the curve

$$t \to (\exp tA)(x_0),\ -\infty < t < \infty,$$

is the (unique) integral curve of $v_A$ passing through the point $x_0$.

3. From exercise 2 deduce that $(\exp sA)(\exp tA) = \exp(s+t)A$.

4. Let $GL(n)$ be the group of invertible $n \times n$ matrices and let $\phi : \mathbb{R} \to GL(n)$ be a homomorphism of the additive group of real numbers into $GL(n)$. Assuming $\phi$ is smooth, prove that there exists a $n \times n$ matrix, $A$, such that $\phi(t) = \exp tA$ for all $t$.

5. Let $A$ and $B$ be $n \times n$ matrices. Prove that the following properties are equivalent:

   (i) $A$ and $B$ commute (as matrices).

   (ii) $\exp tA$ and $\exp sB$ commute for all $s$ and $t$.

   (iii) The Lie bracket of $v_A$ and $v_B$ is zero.

6. Let $A$ be an $n \times n$ matrix. Prove that the following properties are equivalent:

(i) The transpose of $A$ is $-A$.

(ii) $\exp tA$ is in $O(n)$ for all $t \in \mathbb{R}$.

7. Consider the distribution $\mathcal{V}$ in $\mathbb{R}^3$ spanned by

$$V = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x(y+1)\frac{\partial}{\partial z}, \; W = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$$

(i) Show that $\mathcal{V}$ is involutive.

(ii) Consider the projection map $\pi : \mathbb{R}^3 \to \mathbb{R}^2$, $(x, y, z) \mapsto (x, y)$. Show that

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \; Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$$

are the vector fields spanning $\mathcal{V}$ that are $\pi$-related to $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

(iii) Find the integral curves of $X$ and $Y$ respectively.

(iv) What are the integral manifolds of $\mathcal{V}$?

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