MATH4210: Financial Mathematics Tutorial 9

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Continuous Market Models

Question (a)

$\tilde{S}_t = S_t$, $\tilde{\Pi}_t = \Pi_t$

We consider a continuous time market, where the interest rate $r = 0$, and the risky asset $S = (S_t)_{0 \le t \le T}$ follows the Balck-Scholes model with initial value $S_0 = 1$, drift μ and volatility $\sigma > 0$ (without any dividend), so that

$$S_t = S_0 \exp(\mu - \sigma^2/2)t + \sigma B_t$$

$$dS_t = \mu S_t \, dt + \sigma S_t \, dB_t \quad --- (*).$$

Solve the following questions: Slides of B.

(a) A self-financing portfolio is given by (x, ϕ), where x represents the initial wealth of the portfolio, and ϕ_t represents the number of risky asset in the portfolio at time t. Let $\Pi_t^{x,\phi}$ be the wealth process of the portfolio, write down the dynamic of $\Pi^{x,\phi}$ in $t \in [0, T]$ in form of

$$d\Pi_t^{x,\phi} = \alpha_t dt + \beta_t dB_t.$$

Find α and β.

Recall. π_t is a self-financing portfolio

then $d\pi_t = (\pi_t - \phi_t S_t) r\,dt + \phi_t dS_t$

$$\Leftrightarrow \quad d\hat{\pi}_t = \phi_t d\hat{S}_t$$

where $\hat{\pi}_t := e^{-rt}\pi_t$, $\hat{S}_t = e^{-rt} S_t$.

A short proof of the above equivalence:

$$d\hat{\pi}_t = d(e^{-rt}\pi_t) \overset{\text{product rule}}{=} -re^{-rt}\pi_t dt + e^{-rt} d\pi_t$$

$$= -re^{-rt}\cancel{\pi_t} dt + e^{-rt}(\cancel{\pi_t} - \phi_t S_t) r dt + e^{-rt}\phi_t dS_t$$

$$= \phi_t(-re^{-rt}S_t + e^{-rt} dS_t)$$

$$= \phi_t\, d(e^{-rt}S_t)$$

product rule

$$d(e^{-rt}S_t) = -re^{-rt}S_t dt + e^{-rt} dS_t.$$

$$= \phi_t d\tilde{S}_t$$

in our case. $\hat{\pi}_t = \pi_t$, $\hat{S}_t = S_t$ since $r=0$.

$$\Rightarrow d\hat{\pi}_t = \phi_t d\hat{S}_t \Leftrightarrow d\pi_t = \phi_t dS_t$$

By $(*)$. $d\pi_t = \phi_t(\mu S_t dt + \sigma S_t dB_t)$

$$= \mu\phi_t S_t dt + \sigma\phi_t S_t dB_t.$$

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Question (b)

(b) There exists a unique risky-neutral probability $\mathbb{Q}$, together with a Brownian motion $B^{\mathbb{Q}}$ under the probability measure $\mathbb{Q}$. Give the expression of S_t as a function of $(t, B_t^{\mathbb{Q}})$.

$$\left(dS_t \overset{\mathbb{Q}}{=} rS_t\,dt + \sigma S_t\,dB_t^{\mathbb{Q}} = \sigma S_t\,dB_t^{\mathbb{Q}} \right)$$

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma B_t^{\mathbb{Q}}\right)$$

$$\boxed{S_t = \exp\left(-\frac{\sigma^2}{2}t + \sigma B_t^{\mathbb{Q}}\right)}$$

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since $S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma B_t^{\mathbb{Q}} \right)$

Question (c)

(c) We first consider a derivative option with payoff $g(S_T) = S_T^2$ at maturity T.
(i) Compute the value

$$V_0 = \mathbb{E}^{\mathbb{Q}}[S_T^2]. = \mathbb{E}^{\mathbb{Q}}[e^{-rT} g(S_T) \mid S_0]$$

= option price at time 0

$$\mathbb{E}^{\mathbb{Q}}[S_T^2] = \mathbb{E}^{\mathbb{Q}}\left[S_0^2 \exp\left(2\left(\overset{(r=0)}{\cancel{r}} - \frac{\sigma^2}{2}\right)T + 2\sigma B_T^{\mathbb{Q}} \right) \right]$$

$$= e^{(\underset{(r=0)}{\cancel{2r}} - \sigma^2)T}\, \mathbb{E}^{\mathbb{Q}}\left[e^{2\sigma B_T^{\mathbb{Q}}} \right]$$

Recall that $B_T^{\mathbb{Q}} \sim N(0, T)$.

By the characteristic function

$$\mathbb{E}^{\mathbb{Q}}[e^{2\sigma \cdot B_T^{\mathbb{Q}}}] = e^{\frac{1}{2}\cdot(2\sigma)^2 \cdot T} = e^{2\sigma^2 T}$$

$$\Rightarrow \mathbb{E}^{\mathbb{Q}}[S_T^2] = e^{(\cancel{2}r-\sigma^2)T} \cdot e^{2\sigma^2 T}$$

$(r=0)$

$$= e^{(\cancel{2}r+\sigma^2)T}.$$

$$= e^{\sigma^2 T}$$

$\partial_t v(t,x) = -\sigma^2 \boxed{x^2 \exp(\sigma^2(T-t))} = -\sigma^2 V(t,x)$

Question (c)

(ii) Let $v(t,x) := x^2 \exp \sigma^2(T-t)$*, compute* $\partial_t v$*,*$\partial_x v$ *and* $\partial^2_{xx} v$*. Check that* v *satisfies the equation*

$$\partial_t v(t,x) + \frac{1}{2}\sigma^2 x^2 \partial^2_{xx} v(t,x) = 0, \qquad \underline{v(T,x) = x^2.}$$

$\partial_x V(t,x) = 2x \exp(\sigma^2(T-t))$

$\partial^2_{xx} V(t,x) = 2 \exp(\sigma^2(T-t))$

$V(T,x) = x^2 \underbrace{\exp(\sigma^2(T-T))}_{=1}$

$= x^2$

$\partial_t V(t,x) + \frac{1}{2}\sigma^2 x^2 \partial^2_{xx} V(\ ) = -\sigma^2 V(t,x) + \sigma^2 v(t,x) = 0.$

Continuous Market Models

$f \in C^{1,2}(\mathbb{R}_+, \mathbb{R})$

$d f(t, B_t) = \partial_t f(t,B_t)\, dt + \frac{1}{2}\partial^2_{xx} f(t,B_t)\, dt + \partial_x f(t,B_t)\, dB_t$

Question (c)

(iii) Remember that S_t is a function of (t, B_t), apply the Ito formula on $v(t, S_t)$ to deduce that

$S_t = S_0 \exp\left((r - \frac{\sigma^2}{2}) t + \sigma B_t^{\theta} \right)$

$$S_T^2 = V_0 + \int_0^T \phi_t dS_t, \quad where \ \ \phi_t := \partial_x v(t, x).$$

Then deduce that V_0 is the (no-arbitrage) price of the derivative option $g(S_T) = S_T^2$.

$dV(t, S_t) = \partial_t V(t, S_t) + \partial_x V(t, S_t)\, dS_t + \frac{1}{2}\partial^2_{xx} V(t, S_t)\, \underline{\sigma^2 S_t^2\, dt}.$

$= \underline{-\sigma^2 S_t^2 \exp(\cancel{\sigma^2(T-t)})\, dt}$

$$+ 2S_t \exp(\sigma^2(T-t))\, dS_t$$

$$+ \frac{1}{2} \cdot 2\exp(\sigma^2(T-t)) \cdot \sigma^2 S_t^2\, dt$$

$dS_t = \cdots dt + \sigma S_t dB_t$
then
$d\langle S\rangle_t = \sigma^2 S_t^2\, dt$

$$dV(t, S_t) = \partial_t V(t, S_t)\,dt + \frac{1}{2}\partial_{xx}^2 V(t, S_t) \cdot d\langle S\rangle_t + \partial_x V(\ )\,dt.$$

$$dS_t = rS_t\,dt + \sigma S_t\, dB_t$$

$$\text{"}\, dS_t \cdot dS_t = \cancel{(dt)^2} + \cancel{dt\, dB_t} + (dB_t)^2.$$

$$= \sigma^2 \cdot S_t^2 \cdot (dB_t)^2$$

$$= \sigma^2 S_t^2 \cdot dt \text{ "}$$

$$V(T, S_T) = V_0 + \int_0^T 2S_t \exp(\sigma^2(T-t))\, dS_t$$

$$\Rightarrow \quad \phi_t = 2S_t \exp(\sigma^2(T-t)).$$

go back to slides 4B.

X_t, Y_t

$$d(X_t Y_t) = X_t\, dY_t + Y_t\, dX_t + d\langle X, Y\rangle_t.$$

If Y_t is deterministic,

then $d(X_t Y_t) = X_t\, dY_t + Y_t\, dX_t$.

For example: $d(e^{-rt} B_t) = -re^{-rt} B_t + e^{-rt} dB_t.$

since e^{-rt} is deterministic

Back to the original question: (Method 2)

$$V(t,S_t) = V\left(t, S_0 \exp\left(-\frac{\sigma^2}{2}t + \sigma B_t\right)\right)$$

$$= \exp\left(-\frac{\sigma^2}{2}t + \sigma B_t\right)^2 \exp(\sigma^2(T-t))$$

$$= \exp(-\sigma^2 t + 2\sigma B_t + \sigma^2(T-t))$$

$$= \exp(\sigma^2 T)\cdot\exp(-2\sigma^2 t + 2\sigma B_t) = u(t, B_t)$$

Then, by Itô's formula:

$$dV(t,S_t) = d\,u(t,B_t)$$

$$= \exp(\sigma^2 T)\cdot\Big(-2\sigma^2 \cancel{\exp(-2\sigma^2 t + 2\sigma B_t)}\,dt + 2\sigma\exp(-2\sigma^2 t + 2\sigma B_t)\,dB_t$$
$$+ \frac{1}{2}\cdot 4\sigma^2 \cancel{\exp(-2\sigma^2 t + 2\sigma B_t)\,dt}\Big)$$

$$= 2\sigma \exp(\sigma^2(T-t))\cdot\exp(-\sigma^2 t + 2\sigma B_t)\,dB_t$$

Note that $dS_t = rS_t\,dt + \sigma S_t\,dB_t$

$$= \sigma\exp\left(-\frac{\sigma^2}{2}t + \sigma B_t\right)dB_t$$

So $dV(t,S_t) = 2\exp(\sigma^2(T-t))\underbrace{\exp\left(-\frac{\sigma^2}{2}t + \sigma B_t\right)}_{S_t}\cdot\underbrace{\sigma\cdot\exp\left(-\frac{\sigma^2}{2}t + \sigma B_t\right)dB_t}_{dS_t.}$

$$= 2S_t\exp(\sigma^2(T-t))\,dS_t$$

$$= \partial_x V(t,S_t)\,dS_t.$$

So $\int_0^T dV(t,S_t) = \int_0^T \partial_x V(t,S_t)\,dS_t$

$\iff V(T,S_T) - V(0,S_0) = \int_0^T \partial_x V(t,S_t)\,dS_t$

$\iff V(T,S_T) = V_0 + \int_0^T \partial_x V(t,S_t)\,dS_t$

Continuous Market Models

Question (d)

(d) We now consider another option with (path-dependent) payoff $\boxed{\int_0^T S_t^2 dt}$.
(i) Remember that S_t *is a function of* (t, B_t), *apply the Ito formula to deduce that*

$$S_T^2 = S_0^2 + \int_0^T 2S_t dS_t + \sigma^2 \int_0^T S_t^2 dt.$$

$$S_t = S_0 \exp\left((r - \frac{\sigma^2}{2}) t + \sigma B_t^{\theta} \right)$$

$$= \exp\left(-\frac{\sigma^2}{2} t + \sigma B_t^{\theta} \right).$$

$$S_t^2 = \boxed{\exp(-\sigma^2 t + 2\sigma B_t)} = u(t, B_t)$$

$dS_t = \cdots dt + \sigma S_t dB_t$

so

$d\langle S\rangle_t = \sigma^2 S_t^2 dt$

$$d(S_t^2) = 2S_t dS_t + \frac{1}{2}\cdot 2\, d\langle S\rangle_t$$

$$= 2S_t dS_t + \frac{1}{2}\cdot 2\sigma^2 \cdot S_t^2 \cdot dt.$$

$$\Rightarrow S_T^2 - S_0^2 = \int_0^T 2S_t dS_t + \int_0^T \frac{1}{2}\cdot 2\sigma^2 S_t dt$$

$$= \int_0^T 2S_t dS_t + \sigma^2\int_0^T S_t dt$$

$$\Rightarrow S_T^2 = S_0^2 + \int_0^T 2S_t dS_t + \sigma^2\int_0^T S_t dt.$$

Method 2:

$$S_t^2 = \exp(-\sigma^2 t + 2\sigma B_t)$$

$$=: W(t, B_t)$$

By Itô's formula:

$$dW(t, B_t) = -\sigma^2 \underbrace{\exp(-\sigma^2 t + 2\sigma B_t)}_{\partial_t W} dt + \underbrace{2\sigma \exp(-\sigma^2 t + 2\sigma B_t)}_{\partial_x W} dB_t$$

$$+ \frac{1}{2}\cdot \underbrace{4\sigma^2 \exp(-\sigma^2 t + 2\sigma B_t)}_{\partial_{xx}^2 W} dt$$

$$= \sigma^2 \exp(-\sigma^2 t + 2\sigma B_t) dt + 2\sigma \exp(-\sigma^2 t + 2\sigma B_t) dB_t$$

Since $S_t^2 = \exp(-\sigma^2 t + 2\sigma B_t)$, $dS_t = \sigma S_t dB_t$

$$\text{So } dW(t, B_t) = \sigma^2 S_t^2 dt + 2\sigma S_t^2 dB_t$$

$$= \sigma^2 S_t dt + 2S_t dS_t$$

$$\Rightarrow \underset{S_T^2}{\underset{\parallel}{W(T, B_T)}} = \underset{S_0^2}{\underset{\parallel}{W(0, B_0)}} + \sigma^2\int_0^T S_t dt + \int_0^T 2S_t dt$$

Continuous Market Models

$S_T^2 = S_0^2 + \int 2S_t dS_t + \sigma^2 \int S_t dt$

Question (d)

(ii) From the above, one obtains that

$$\sigma^2 \int_0^T S_t^2 dt = S_T^2 - S_0^2 - \int_0^T 2S_t dS_t.$$

Deduce the replication cost and replication strategy of the derivative option $\int_0^T S_t^2 dt$. (Hint: Use the above replication strategy for the option $g(S_T) = S_T^2$.)

Π_t replicate the option with payoff $\int_0^T S_t^2 dt$:

i.e. $\forall t \in [0, T]$, $\Pi_t = E^{\mathbb{Q}}[\int_0^T S_u^2 du \mid S_t]$

So $\Pi_T = \int_0^T S_t^2 dt$

$$\int_0^T S_t^2\,dt = \Pi_T = \frac{S_T^2 - S_0^2}{\sigma^2} - \frac{1}{\sigma^2}\int_0^T \underline{2S_t}\,dS_t$$

In (c) (iii), $S_T^2 = V_0 + \int_0^T \phi_t\,dS_t$

$$= V_0 + \int_0^T 2S_t \exp(\sigma(T-t))\,dS_t$$

So: $$\int_0^T S_t^2\,dt = \frac{1}{\sigma^2}\left(V_0 + \int_0^T \phi_t\,dS_t - S_0^2\right) - \frac{1}{\sigma^2}\int_0^T 2S_t\,dS_t$$

$$= \frac{1}{\sigma^2}\left(e^{\sigma^2 T} - S_0^2\right) + \frac{1}{\sigma^2}\int_0^T \phi_t\,dS_t - \frac{1}{\sigma^2}\int_0^T 2S_t\,dS_t$$

$$= \frac{1}{\sigma^2}\left(e^{\sigma^2 T} - 1\right) + \frac{1}{\sigma^2}\int_0^T \psi_t\,dS_t$$

where $\psi_t := \phi_t - 2S_t$

Therefore, $\Pi_T = \Pi_0 + \int_0^T \frac{1}{\sigma^2}\psi_t\,dS_t$ defines the replicating portfolio of option with payoff $\int_0^T S_t^2\,dt$.

Hence, the price of the option at time $t=0$ is $\frac{1}{\sigma^2}\left(e^{\sigma^2 T} - 1\right)$

Check: $\mathbb{E}^{\mathbb{Q}}\left[\int_0^T S_t^2\,dt \,\middle|\, S_0 = 1\right]$

$$= \int_0^T \mathbb{E}^{\mathbb{Q}}[S_t^2]\,dt$$

$$= \int_0^T \mathbb{E}^{\mathbb{Q}}\left[\exp(-\sigma^2 t + 2\sigma B_t)\right]dt$$

$$= \int_0^T \exp(-\sigma^2 t) \cdot e^{\frac{1}{2} \cdot 4\sigma^2 \cdot t} \, dt$$

$$= \int_0^T \exp(\sigma^2 t) \, dt$$

$$= \frac{1}{\sigma^2} \Big[\exp(\sigma^2 t) \Big]_0^T$$

$$= \frac{1}{\sigma^2} \left(e^{\sigma^2 T} - 1 \right)$$

Remark for method 1.

If S_t follows the dynamic:

$$dX_t = \mu X_t \, dt + \sigma X_t \, dB_t$$

Then: (Itô's formula): for $f \in C^{1,2}(\mathbb{R}_+, \mathbb{R})$

$$df(t, X_t) = \partial_t f(t, X_t) dt + \partial_x f(t, X_t) \cdot dX_t + \frac{1}{2} \partial^2_{xx} f(t, X_t) \cdot \sigma^2 X_t^2 \, dt$$

$$= \partial_t f(t, X_t) dt + \mu X_t \partial_x f(t, X_t) \, dt + \sigma X_t \partial_x f(t, X_t) dB_t + \frac{1}{2} \sigma^2 X_t^2 \partial^2_{xx} f(t, X_t) \, dt$$