# MATH4210: Financial Mathematics Tutorial 7

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# Stochastic Integration

## Question

*For fixed $T > 0$, prove that*

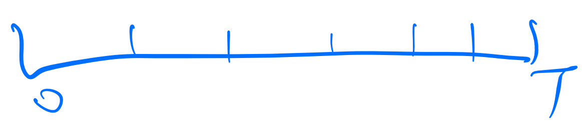
$$\int_0^T B_t dBt = \frac{1}{2}B_T^2 - \frac{1}{2}T$$

*from sketch. $(B_t)_{t\geq 0}$ is a standard Brownian motion.*

Recall, $\int_0^T \theta_t dB_t = \lim_{n\to+\infty} \sum_{k=0}^{n-1} \theta_{t_k}(B_{t_{k+1}} - B_{t_k})$. where $t_{k+1} - t_k = \frac{T}{n}$.

Proof: Define for $n > 0$, $\Delta t = \frac{T}{n}$, $\theta_t^n = B_{t_k}$ for $t \in [t_k, t_{k+1}]$

where $t_k = \frac{T}{n} \cdot k$.



① $\int_0^T \theta_t^n \, dB_t \to \int_0^T B_t \, dB_t$ as $n \to +\infty$.

Consider $\mathbb{L}^2$ - convergence of the above estimation.

$\Leftrightarrow$ Consider $\mathbb{H}^2[0,T]$ - convergence of $\theta_t^n \to B_t$

i.e. $\mathbb{E}\left[\int_0^T (\theta_t^n - B_t)^2 dt\right]$

$$= \mathbb{E}\left[\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (B_{t_k} - B_t)^2 dt\right]$$

$$= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E}[(B_t - B_{t_k})^2] \, dt$$

$B_t - B_{t_k} \sim N(0, t - t_k)$

$$= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t - t_k) \, dt$$

$$\leq \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \frac{T}{n} \, dt$$

$$= \frac{T^2}{n}. \to 0 \quad \text{as } n \to +\infty.$$

② $\because \int_0^T B_t \, dB_t = \lim_{n \to +\infty} \int_0^T \theta_t^n \, dB_t$

$$= \lim_{n \to +\infty} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} B_{t_k} \cdot dB_t$$

$$= \lim_{n \to +\infty} \sum_{k=0}^{n-1} \cdot B_t \cdot (B_{t_{k+1}} - B_{t_k}).$$

$$= \lim_{n \to +\infty} \sum_{k=0}^{n-1} \left[\underbrace{\frac{1}{2}(B_{t_{k+1}}^2 - B_{t_k}^2)}_{I_1} - \underbrace{\frac{1}{2}(B_{t_{k+1}} - B_{t_k})^2}_{I_2}\right]$$

$I_1 = \frac{1}{2}(B_T^2 - B_0^2) = \frac{1}{2} B_T^2$

$I_2 = \frac{1}{2} \sum_{k=0}^{n-1} (B_{t_{k+1}} - B_{t_k})^2 \xrightarrow{\mathbb{L}^1} \frac{T}{2} \Rightarrow$ subsequence cv. a.s. to $\frac{T}{2}$

Therefore $\int_0^T B_t \, dB_t = \frac{1}{2} B_T^2 - \frac{T}{2}$ ※

# PDEs

Use page 18, 19. of slides 2.

## Question

*Assume u satisfies*

$$\begin{cases} \partial_t u + \frac{\sigma^2}{2} S^2 \partial^2_{SS} u + rS\partial_S u - ru = 0 \\ u(T, S) = g(S) \end{cases}$$

*Compute* $u(0, S_0)$

proof: consider $S = e^x \Leftrightarrow x = \ln S$.

consider $V(t, x) = u(t, e^x)$

$\partial_t V = \partial_t u$.

$\partial_x V = e^x \partial_S u = S \partial_S u$.

$$\partial_{xx}^2 V = e^{x}\partial_s u + e^{2x}\partial_{ss}^2 u.$$

$$= S\partial_s u + S^2\partial_{ss}^2 u \quad \Rightarrow S^2\partial_{ss}^2 u = \partial_{xx}^2 V - S\partial_s u.$$
$$= \partial_{xx}^2 V - \partial_x V$$

$$\Rightarrow \begin{cases} \partial_t V + \frac{\sigma^2}{2}(\partial_{xx}^2 V - \partial_x V) + r\partial_x V - rV = 0 \\ V(T, x) = g(e^x). \end{cases} \quad (*).$$

Consider $W(t,x) = e^{-rt}V(t,x)$.

$$\partial_t W = -re^{-rt}V + e^{-rt}\partial_t V. \Rightarrow -rV + \partial_t V = e^{rt}W$$

$$\partial_x W = e^{-rt}\partial_x V \quad \Rightarrow e^{rt}\partial_x W = \partial_x V$$

$$\partial_{xx}^2 W = e^{-rt}\partial_{xx}^2 V \quad \Rightarrow e^{-rt}\partial_{xx}^2 W = \partial_{xx}^2 V$$

$(*)$ becomes:

$$\begin{cases} \cancel{e^{rt}}\left(\partial_t W + \frac{\sigma^2}{2}(\partial_{xx}^2 W - \partial_x W) + r\partial_x W\right) = 0 \\ W(T,x) = e^{-rT}g(e^x). \end{cases}$$

$$\Leftrightarrow \begin{cases} \partial_t W + \underline{(r - \frac{\sigma^2}{2})\partial_x W} + \frac{\sigma^2}{2}\partial_{xx} W = 0 \\ W(T,x) = e^{-rT}g(e^x) \end{cases} \quad (**)$$

By Sides 2. p. 18–19, there exists a generalize BM.

$$X_T = X_0 + (r - \frac{\sigma^2}{2})T + \sigma B_T.$$

s.t. $W(t,x) = \mathbb{E}[e^{-rT}g(X_T) \mid X_t = x]$ solves the Heat Equation $(**)$.

Note That $W(0, X_0) = V(0, X_0) = u(0, e^{X_0}) = u(0, S_0)$

So $u(0, S_0) = W(0, X_0) = \mathbb{E}[e^{-rT}g(X_T) \mid X_0 = x]$

$$= \mathbb{E}[e^{-rT}g(e^{X_0 + (r-\frac{\sigma^2}{2})T + \sigma B_T})]$$

$$= \mathbb{E}[e^{-rT}g(S_0 e^{(r-\frac{\sigma^2}{2})T + \sigma B_T})]$$

# Ito Formula

Recall for $f \in C^{1,2}(\mathbb{R}^+, \mathbb{R})$.

$$f(t, B_t) = f(0, B_0) + \int_0^t (\partial_t f + \frac{1}{2} \partial_{xx}^2 f)(s, B_s)\, ds + \int_0^t \partial_x f(s, B_s)\, dB_s .$$

## Question

*Consider a standard Brownian motion $(B_t)_{t\geq 0}$. Let $T > 0$, compute*
*(a)* $\int_0^T B_t dBt$
*(b)* $\int_0^T \exp(B_t - \frac{1}{2}t)dt$
*using Ito formula.*

(a). Consider $f : x \mapsto x^2$, $f'(x) = 2x$, $f''(x) = 2$.

By Itô's formula,

$$f(B_T) = f(B_0) + \frac{1}{2}\int_0^T f''(B_s) ds + \int_0^T f'(B_s)\, dB_s,$$

$$\Leftrightarrow B_T^2 = B_0^2 + \cancel{\frac{1}{2}} \int_0^T \cancel{2} \cdot ds + \int_0^T 2 B_s dB_s .$$

$$\Leftrightarrow \int_0^T B_s\,dB_s = \frac{1}{2}B_T^2 - \frac{1}{2}T$$ ※

(b). Consider $f : (t,x) \mapsto e^{x-\frac{1}{2}t}$.

$$\partial_t f = -\frac{1}{2} f \quad , \quad \partial_x f = f \quad , \quad \partial_{xx}^2 = f \, ,$$

$$f(T, B_T) = f(0, B_0) + \int_0^T (\partial_t f + \frac{1}{2}\partial_{xx}^2 f)(s, B_s)\,ds$$

$$+ \int_0^T \partial_x f(s, B_s)\,dB_s$$

$$\exp(B_T - \tfrac{1}{2}T) = 1 + \int_0^T \exp(B_s - \tfrac{1}{2}s)\,dB_s$$

$$\Rightarrow \int_0^T \exp(B_t - \tfrac{1}{2}t)\,dB_t = \exp(B_T - \tfrac{1}{2}T) - 1$$ #

$$f(t, B_t) = \exp(B_t - \tfrac{1}{2}t).$$

$$df(t, B_t) = \cancel{-\frac{1}{2}\exp(B_t - \tfrac{1}{2}t)\,dt + \frac{1}{2}\exp(B_t - \tfrac{1}{2}t)\,dt}$$

$$\| $$

$$d(\exp(B_t - \tfrac{t}{2})). \quad + \exp(B_t - \tfrac{1}{2}t)\,dB_t$$

$$\int_0^T \underset{1}{} d(\exp(B_t - \tfrac{t}{2})) = \int_0^T \exp(B_t - \tfrac{1}{2}t)\,dB_t$$

$$\exp(B_T - \tfrac{T}{2}) - 1 = \int_0^T \exp(B_t - \tfrac{1}{2}t)\,dB_t.$$