MATH4210: Financial Mathematics Tutorial 11

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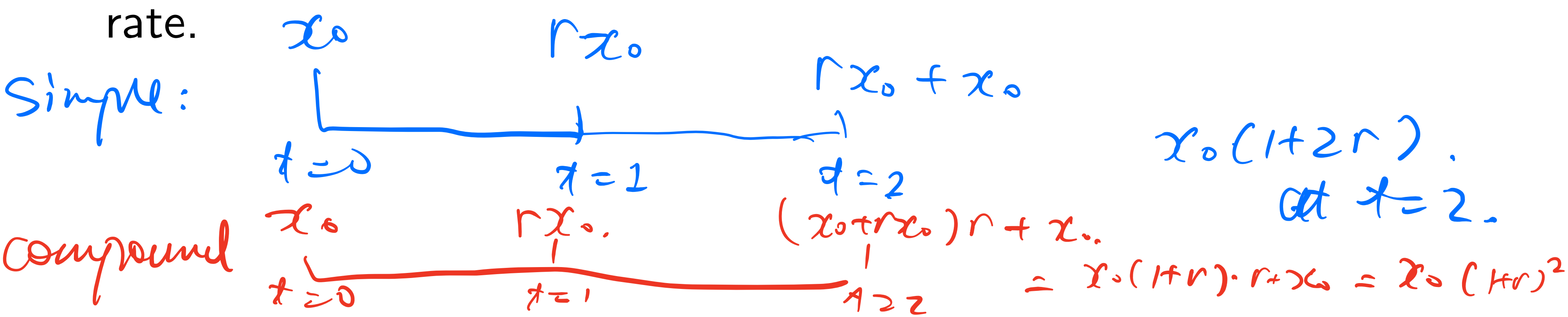
Interest Rate

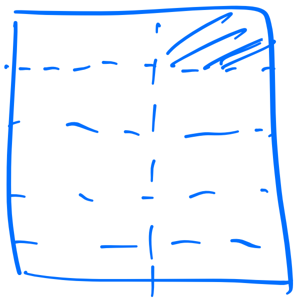
principal: x_0.

Let r be the interest rate. Suppose that you place \$$x_0$ in an account in a bank. After n years, you will have the amount

- $y_n = x_0(1 + nr)$ if the interest rate is the simple interest rate.
- $y_n = x_0(1 + r)^n$ if the interest rate is the annual compound interest rate. reinvest your interest rate into the bank.
- $y_n = x_0(1 + \frac{r}{m})^{mn}$ if the interest rate is the compound interest rate and compound m times per annul. { semi annually ... / quarterly ...
- $y_n = x_0 e^{nr}$ if the interest rate is the continuous compound interest rate.



Interest Rate



face value \$100. coupon 1\$ 5 year.

Bond $\Leftrightarrow$ price the interest rate

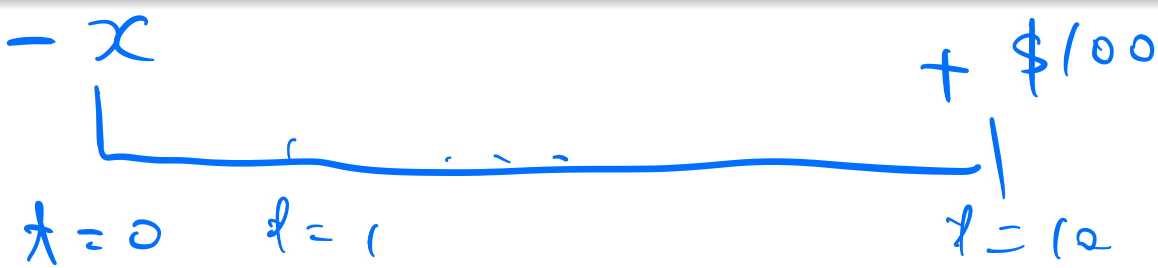
Options $\Leftrightarrow$ price for volatility of S_t

Question

$(dS_t = \mu S_t dt + \sigma S_t dB_t)$

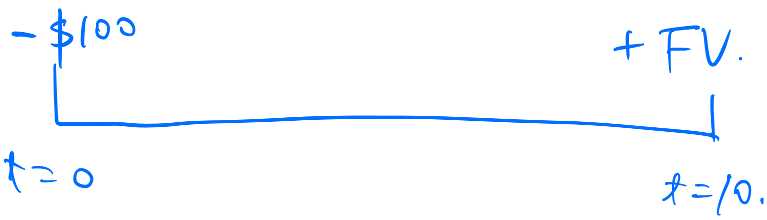
a) Find the value of a 10-year zero-coupon bond of face value \$100 *if the the annual simple interest rate is* 2%.

b) Find the face value of a 10-year zero-coupon bond if it is issued for \$100 *and the continuous compound interest rate is* 3%.



$$x = \frac{\$100}{(1+10r)}.$$

discounted future cash flow is the value of the product (b).



$$FV \cdot e^{-10r} = \$100$$

$$PV = \sum \text{discount factor} \times \text{Cash flow.}$$

$$PV = \sum_{i=0}^{n} \frac{CF_i}{(1+ir)}$$ simple

$$PV = \sum_{i=0}^{n(+\infty)} \frac{CF_i}{(1+r)^i}$$ compound annually.

Present Value

Since we can always use $\$x_0$ now as principal in a risk-free investment at (continuous compound interest) rate $r > 0$ guaranteeing the amount

$$x_0 e^{rt} > x_0$$

at time t. Equivalently, if we deposit $\$xe^{-rt}$ at the bank, we get $\$x$ at time t, thus

We call xe^{-rt} the present value (PV) of x,

which is also called the discounted value of x at the future time t, and the factor e^{-rt} is called the discount factor.

Present Value

$-PV$ $+50$ $+50$ $+50$ $+50+2000$

$t=0$ $t=0.5$ $t=1$ $t=1.5$ $t=2$

Question

par value (face value)

Pricing a coupon bond: consider a 2-year \$2000 bond, that has coupons every 1/2 year in the amount of \$50, for a total of four times until 2 years at which time you receive \$2050. Suppose the continuous compound interest rate is r. What is its price of the bond?

$$PV=\sum_{n=1}^{4} 50\cdot e^{-r\cdot 0.5n}+2000e^{-r\cdot 2}$$

Annuity/Perpetual Bond

Question

An imaginary nice government that does not exists on this planet promises to pay you (and your descendant) \$10,000 *immediately and the same amount every year perpetually. If the the compound annual interest rate is* 2.5%, *what is its present value of this plan?*

$CF_i = \$10{,}000. \quad \forall\, i \geq 0$

discount $CF_i = \$10000 \times \frac{1}{(1+2.5\%)^i}$

Then $PV = \sum_{i=0}^{+\infty} \$10000 \cdot \frac{1}{(1+2.5\%)^i} = 10000 \cdot \frac{1}{1-\frac{1}{1.025}}$

Annuity/Perpetrial Bond

Plan 1: PV_1, $CF_i = -1000$, $\forall i \geq 1$. perpetual bond

plan 2: $PV_1 = -50000$, $CF_i = 0$. $\forall i \geq 1$.

Question

Joyce wants to use a land to build a church. The government requires ~~she~~ her to pay the nominal rent 1,000 *every year perpetually. A bank offers a plan: Joyce pay the bank* 50,000 *at once and the bank promises to pay* 1,000 *to the government every year. Suppose the discrete annual compound interest is* 2%. *Should Joyce accept this offer? (Unit: \$)*

$$PV_1 = \sum_{i=1}^{+\infty} \frac{(1000)}{(1+2\%)^i}$$

compare with PV_2.

if $PV_1 \geq PV_2$ accept.

Options Revisit

Question (Example on Slides 5)

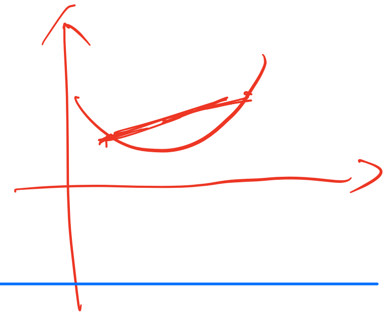
(a).Suppose that we have three European call options with the same maturity T in the financial market whose price at time $t = 0$ are:

$$C_1(K = 90) = 10 \qquad T = 2$$
$$C_2(K = 100) = 9 \qquad P(K=100) = 9. \quad T = 1$$
$$C_3(K = 110) = 7. \qquad T = 2.$$

Suppose the interest rate is zero. Construct the arbitrage strategy.
(b). At $t = 0$, the underlying asset $S_0 = 100$. We keep C_1 and C_3 the same. But We don't have C_2, instead there is a European put option with the same setting such that $P_2(K = 100) = 9$. Find the arbitrage strategy.

For call options with all setting the same except the strike price.

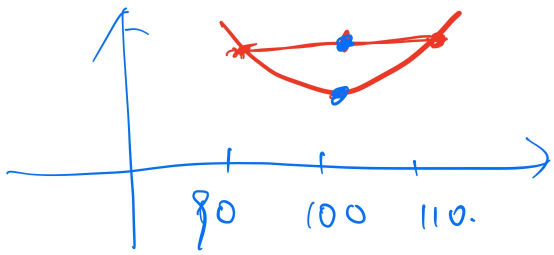
$K \mapsto C(K)$ is convex



$C_1(K=80) = 10$

$C_2(K=100) = 8$

$C_3(K=110) = 7$.



Assume C_1, C_3 are well-priced.

Then. $C_1 + C_3 = 17$ should be greater or equal to $2 \cdot C(K=100)$.

$\Rightarrow C(K=100) \leq 8.5$

However $C_2(K=100) = 8 > 8.5$.

$\Rightarrow C_2$ is higher than the actual value.

$\Pi(t) = \boxed{C_1 + C_3} - \boxed{2C_2}$.

(b) ① use convexity of C to estimate $C(K=100)$.

② use put-call parity to get $P(K=100)$