Choice-Based Cluster Consensus in Multi-Agent Systems

LIU Zhongchang, WONG Wing Shing

Department of Information Engineering, The Chinese University of Hong Kong, Shatin, N. T., Hong Kong.
E-mail: zcliu@ie.cuhk.edu.hk, wswong@ie.cuhk.edu.hk

Abstract: In this paper, the cluster consensus problem for multiple agents whose state evolution are described by identical linear models is investigated. In this model, which is generic, every agent only observes relative state information from exactly one neighbor. Control decisions for each agent are dependent on the observed data and a parameter chosen at an initial time from a set of possible options. These options are referred to as agent choices. The observation pattern of the agents is summarized by an interacting graph. The key result of this paper shows that under suitable structural conditions of the interacting graph, there exists a choice-based distributed control algorithm that enables agents who make the same choices to converge to a common consensus state, while agents who make different choices will converge to distinct states. In additional to graph topology, the structural condition also depends on how the choice selections are distributed. For interacting graphs that satisfy the so called *unicyclic* property, an explicit characterization of the sets of choice selections that guarantee convergence is presented.

Key Words: Multi-agent System, Distributed Choices, Cluster Consensus

1 Introduction

The past decade witnessed prolific development in the study of collective and coordinated behaviors of multiple agents (see the recent survey papers [1, 2] for details.) Most of the literature are focused on studying aggregation phenomena of the agents, such as the consensus problems with leaders [3, 4] or without leaders [5], swarming/flocking [6, 7], and rendezvous problem [8, 9].

It is common to encounter in biological and engineering systems interesting scenarios where a group of agents over time splits into sub-groups. Some obvious examples are: predator evasion and separated foraging for a flock of birds or a herd of animals [10], obstacles avoidance [7] or multiple tasks searching for autonomous vehicles, and heterogeneous robots sorting [11]. In social networks, people with different believes or opinions tend to separate into different sub-groups [12]. These phenomena are commonly labeled as cluster consensus, or clustering, which refers to the evolution of a network of agents into a partition of clusters, with all agents within the same cluster agree upon a common state.

Emerging investigations on clustering can be found in flocking [11, 13, 14] and consensus/sychronization [15–18] problems. In [13, 14], the authors presented a swarm aggregation and splitting control scheme where an intermediate-range Gaussian-type repulsive interaction among agents could split a cohesive group into several sub-groups. In [11] segregation of two types of agents is realized by utilizing differential inter- and intragroup artificial potentials. In consensus/sychronization problems, a basic assumption towards clustering is that all agents in a cluster have the same in-degree from other clusters (common inter-cluster coupling condition). In [15], necessary and sufficient conditions for group consensus are provided for switching multi-agent networks by introducing double-tree-form transformations. The authors in [16] found two inter-cluster and intra-cluster coupling conditions for networks of nonidentical systems to realize cluster synchronization. In [17] there is a report on three different mechanisms that lead to clustering of a multi-agent system: the existence of different self-dynamics, and for agents with identical self-dynamics, the presence of delays and the existence of both positive and negative couplings. In addition, networks of agents with nonlinear self-dynamics [19] or generic linear self-dynamics [18] can be forced into clusters by pinning control techniques.

For tiny, low cost implementation of an agent or for cases where stealthiness is a necessity, active communication is commonly avoided. Hence, there is an emerging interest in studying control algorithms that call for as less communication as possible. The authors in [20, 21] interpreted the complexity of communicating in a control system in terms of the control energy required, where [21] showed that without communication, choice actions in a class of bilinear systems can be implemented at additional control energy cost. In vehicle routing problems [22, 23], tasks can be completed by multiple cooperative agents without explicit information exchange among them. Recently, Baillieul et al [24, 25] tried to use of the relative motion between robots for information signaling.

Besides the many attractive benefits in engineering applications, lack of communication also emerges frequently in engineering systems and social networks. For example, a communication may link fail; nodes in a network may not be fully conscious of the activities of the other nodes; human society is fraught with situations with no or partial people to people communication. In most of these scenarios, every agent may still have an individual option, or choice, and acts based on it and other observed information. Therefore, there is a need to study how agents can split into clusters based on choice and as little inter-agent communication as possible.

In this paper, we consider the clustering phenomenon resulting from distributed choices of interconnected agents in a network. The only information an agent can obtain comes from its direct neighbor; no explicit communication channel exists between any two agents. To reflect this special information coupling pattern, the underlying network topology is assumed to be a directed graph and each agent links to exactly one other agent. Agents make the same choice are equipped with the same choice-based control law. Under some structural conditions of the underlying network, clustering is proved to appear when agents make different

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choices. Explicit characterization of the choice selection distributions that satisfy the structural conditions are presented under the assumed interacting graph structure.

The structure of this paper is as follows: In section 2, we present background information from relevant graph theory and state the problem formulation. In section 3, main results for cluster consensus are derived and characterization of the choice selection distributions leading to convergence is presented. Simulations are conducted in section 4 to verify and demonstrate our results. Conclusions follow in section 5.

2 Preliminary

2.1 Graph theory

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be the topology of a directed graph (or digraph) with node set $\mathcal{V} = \{v_1, \ldots, v_L\}$ and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The nodes are indexed by a finite set $\mathcal{I} = \{1, 2, \ldots, L\}$. A directed edge, or arc, from v_i to v_j is an ordered pair of distinct nodes $(v_i, v_j) \in \mathcal{E}, i, j \in \mathcal{I}, i \neq j$. The set of neighbors of node i is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$. $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{L \times L}$ is the $0-1$ adjacency matrix, i.e., $a_{ij} = 1$ if and only if $(v_j, v_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{L \times L}$ is the Laplacian of $\mathcal{G}$, where $l_{ii} = \sum_{j=1}^{L} a_{ij}$ and $l_{ij} = -a_{ij}$ if $i \neq j$. A directed path in $\mathcal{G}$ is a sequence $v_{i_1}, v_{i_2}, \ldots, v_{i_K}$ of distinct nodes such that $(v_{i_j}, v_{i_{j+1}})$ is an arc of $\mathcal{G}$ for $j = 1, \ldots, K-1$.

A digraph is (strongly) connected if and only if any two distinct nodes can be joined by a (directed) path, and a digraph is weakly connected if and only if its underlying undirected graph is connected. An *unicyclic graph* (for example, Fig.1) is a weakly connected graph where the number of nodes equals the number of edges. A directed tree is a digraph where every node except the root has exactly one parent. A directed spanning tree of a digraph is a directed tree formed by graph edges that connects all the nodes of the digraph. The in-degree of a node in a digraph is the number of arcs ending on that node ([26]).

Now we give a characterization of unicyclic graphs with directed edges.

Proposition 1. *Let $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}}, \mathcal{A})$ be the topology of a directed graph such that the in-degree of every node is* 1. *Then, $\bar{\mathcal{G}}$ is an unicyclic graph contains exactly one directed cycle $\mathcal{C}$ and directed trees attached to the nodes of $\mathcal{C}$.*

Note that there can be several directed trees attached to the same node in the cycle. Also, it is easily seen that $\bar{\mathcal{G}}$ has a directed spanning tree.

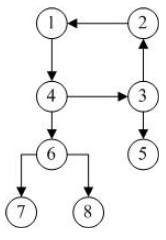

Fig. 1: A sample of a directed graph $\bar{\mathcal{G}}$.

2.2 Problem formulation

Consider a network consisting of L agents described by linear dynamic models:

$$\dot{x}_l(t) = Ax_l(t) + Bu_l(t), \ \ l \in \mathcal{I} = \{1, 2, \ldots, L\}. \quad (1)$$

where $x_l(t) \in \mathbb{R}^n$, $u_l(t) \in \mathbb{R}^m$ are the state and control of the agent l, respectively, and $A \in \mathbb{R}^{n \times n}$ is the system matrix. A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is associated with system (1) such that each agent is regarded as a node and information flow from agent j to agent l corresponds to a directed edge $(v_j, v_l) \in \mathcal{E}$. An agent j is said to be a neighbor of l if and only if (v_j, v_l) is an edge.

In this paper, we are interested in networks of agents where no explicit communication is allowed between agents, owing to reasons such as radio silence, lack of or breakdown of communication links, etc. So, agents interact by observation and it is convenient for each of them to observe only one other agent.

Assumption 1. *Every agent has exactly one neighbor, i.e., the network of L agents has unicyclic graph topology $\bar{\mathcal{G}}$ as described in Proposition 1.*

For agents with generic system dynamics A, we only consider stabilizable systems.

Assumption 2. *The linear system (A, B) is stabilizable.*

Each agent l chooses independently from a choice set labeled by $\Omega := \{1, 2, \cdots\}$ at the initial time and all agents' choices remain unchanged during the time period a task. This choice set is a numeric representation of terms which can be endowed with specific meanings such as opinions, beliefs, or physical states of an agent. Denote by integer $i_l \in \Omega$, the choice outcome (or choice) of agent l. Then, the definition of cluster consensus can be given as follows:

Definition 1 (choice-based cluster consensus). *A network of agents indexed by $\mathcal{I}$ is said to achieve choice-based cluster consensus, if the states of the agents satisfy $\forall l, j \in \mathcal{I}, i_l, i_j \in \Omega$, $\lim_{t\to\infty} \|x_l(t) - x_j(t)\| = 0$, $\forall i_l = i_j$, and $\lim_{t\to\infty} \|(x_l(t) - x_j(t))\| > 0$, $\forall i_l \neq i_j$.*

Remark 1. *The definition differs from those in [15, 17] in that the number of clusters and the members in each cluster are not known as a priori due to different possible combinations of all agents' choices. Also, agents may not go to the same cluster even when they have made the same choice, since very limited information is available for each of them. So, even the agents are separated into several clusters in the sense of [15, 17], it may not be a choice-based cluster consensus.*

For a network of $L \geq 2$ agents with topology $\bar{\mathcal{G}}$ and choice set Ω, this paper devotes to separate these agents into N clusters if there exist N distinct choices, such that each cluster $i, 1 \leq i \leq N$, consists of l_i agents who have made the same choice. So, $l_1 + \cdots + l_N = L$. Note that N is changeable according to the number of choices.

As no direct communication is available, agents cannot exchange their choice information to each other, nor is possible for agents to stabilize themselves to a selected state, since only relative information $x_{lj} := x_l - x_j$ with respect to its neighboring agent j is available to an arbitrary agent l. Therefore, the control input of each agent is a function of its own choice and their relative information, i.e., $u_l = f(i_l, x_{lj})$. In this paper, the following distributed control protocol is proposed: for $l = 1, 2 \cdots L$, $i_l = 1, 2 \cdots, N$, $N \leq L$,

$$u_l^{i_l}(t) = K[x_{j_l}(t) - x_l(t) + h_{i_l}], \tag{2}$$

where j_l is the neighbor of l, K is the controller gain matrix to be designed, and $h_{i_l} \in \mathbb{R}^n$ is a choice based vector such that $h_{i_l} \neq h_{i_m}, \forall i_l \neq i_m, 1 \leq i_l, i_m \leq N$.

3 Main Results

Notations: $\mathbf{1}_n = [1, 1, \ldots, 1]^T \in \mathbb{R}^n$. $I_n \in \mathbb{R}^{n\times n}$ is the n-dimensional identity matrix. $diag\{M_1 \cdots M_n\}$ is the block diagonal matrix constructed from matrices $M_1, \cdots, M_n$. "$\otimes$" stands for the Kronecker product.

3.1 Cluster consensus for graphs with fixed topology

In this section, we will show that when choices are determined and N choices exist, cluster consensus can be achieved by protocol (2). Before deriving our main result, we need the following lemmas.

Lemma 1 ([4, 27]). *$\bar{\mathcal{G}}$ contains a spanning tree, and the Laplacian matrix $\mathcal{L}_{\bar{\mathcal{G}}}$ has a simple eigenvalue 0 with corresponding eigenvector $\mathbf{1}_L$, and the real part of other eigenvalues is positive. Also, $\mathcal{L}_{\bar{\mathcal{G}}}$ has a nonnegative left eigenvector $r \in \mathbb{R}^L$ satisfying $r^T\mathbf{1}_L = 1$, associated with the eigenvalue 0, i.e., $r^T\mathcal{L}_{\bar{\mathcal{G}}} = \mathbf{0}$.*

Definition 2. *A matrix $M \in \mathbb{R}^{nl_1\times nl_2}$ of block matrix form:*

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1l_2} \\ M_{21} & M_{22} & \cdots & M_{2l_2} \\ \vdots & \vdots & \ddots & \vdots \\ M_{l_11} & M_{l_12} & \cdots & M_{l_1l_2} \end{bmatrix}$$

where $M_{pq} \in \mathbb{R}^{n\times n}, p = 1 \cdots l_1, q = 1 \cdots l_2$, is said to have a constant row block-matrix sum $C \in \mathbb{R}^{n\times n}$, if $\sum_{q=1}^{l_2} M_{pq} = C, \forall p$, or $M(\mathbf{1}_{l_2} \otimes I_n) = \mathbf{1}_{l_1} \otimes C$.

Lemma 2. *A square matrix $\bar{M} \in \mathbb{R}^{nL\times nL}$ which is partitioned as:*

$$\bar{M} = \begin{bmatrix} \bar{M}_{11} & \bar{M}_{12} & \cdots & \bar{M}_{1N} \\ \bar{M}_{21} & \bar{M}_{22} & \cdots & \bar{M}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{M}_{N1} & \bar{M}_{N2} & \cdots & \bar{M}_{NN} \end{bmatrix}$$

where $\bar{M}_{ij} = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1l_j} \\ M_{21} & M_{22} & \cdots & M_{2l_j} \\ \vdots & \vdots & \ddots & \vdots \\ M_{l_i1} & M_{l_i2} & \cdots & M_{l_il_j} \end{bmatrix} \in \mathbb{R}^{nl_i\times nl_j}$,
$M_{pq} \in \mathbb{R}^{n\times n}$, for $p = 1 \cdots l_i, q = 1 \cdots l_j$, $1 \leq i, j \leq N$, and $l_1 + \cdots + l_N = L$. If the matrices $\bar{M}_{ij}$, $1 \leq i, j \leq N$, have constant row block-matrix sums $P_{ij} \in \mathbb{R}^{n\times n}$, and $e^{\bar{M}}$ is partitioned in the same way as $\bar{M}$, then the block matrices of $e^{\bar{M}}$ also have constant row block-matrix sums denoted by $Q_{ij} \in \mathbb{R}^{n\times n}$ for $1 \leq i, j \leq N$. In addition, let $P = [P_{ij}]_{N\times N}$ and $Q = [Q_{ij}]_{N\times N}$; then, $Q = e^P$.

Proof. Denote $\bar{D} = diag\{\mathbf{1}_{l_1} \cdots \mathbf{1}_{l_N}\} \otimes I_n$. If $\bar{M}_{ij}$ have constant row block-matrix sums P_{ij} for all $1 \leq i, j \leq N$, then we have

$$\begin{aligned} &\bar{M}(diag\{\mathbf{1}_{l_1} \cdots \mathbf{1}_{l_N}\} \otimes I_n) = \bar{M}\bar{D} \\ =&\begin{bmatrix} \bar{M}_{11}(\mathbf{1}_{l_1} \otimes I_n) & \cdots & \bar{M}_{1N}(\mathbf{1}_{l_N} \otimes I_n) \\ \vdots & \ddots & \vdots \\ \bar{M}_{N1}(\mathbf{1}_{l_1} \otimes I_n) & \cdots & \bar{M}_{NN}(\mathbf{1}_{l_N} \otimes I_n) \end{bmatrix} \\ =&\begin{bmatrix} \mathbf{1}_{l_1} \otimes P_{11} & \cdots & \mathbf{1}_{l_1} \otimes P_{1N} \\ \vdots & \ddots & \vdots \\ \mathbf{1}_{l_N} \otimes P_{N1} & \cdots & \mathbf{1}_{l_N} \otimes P_{NN} \end{bmatrix} = \bar{D}P \end{aligned}$$

This implies $\bar{M}^k\bar{D} = \bar{D}P^k$. It follows that

$$\begin{aligned} e^{\bar{M}}\bar{D} =&(I_{nL} + \bar{M} + \frac{1}{2!}\bar{M}^2 + \cdots)\bar{D} \\ =&\bar{D} + \bar{D}P + \frac{1}{2!}\bar{D}P^2 + \cdots \\ =&\bar{D}e^P. \end{aligned} \tag{3}$$

Partitioning $e^{\bar{M}} = [\Phi_{ij}]_{N\times N}$ in the same way as $\bar{M}$, where $\Phi_{ij} \in \mathbb{R}^{nl_i\times nl_j}$ for $1 \leq i, j \leq N$, and partitioning $e^P = [\Psi_{ij}]_{N\times N}$ in the same way as P, where $\Psi_{ij} \in \mathbb{R}^{n\times n}$ for $1 \leq i, j \leq N$, we have

$$e^{\bar{M}}\bar{D} = \begin{bmatrix} \Phi_{11}(\mathbf{1}_{l_1} \otimes I_n) & \cdots & \Phi_{1N}(\mathbf{1}_{l_N} \otimes I_n) \\ \vdots & \ddots & \vdots \\ \Phi_{N1}(\mathbf{1}_{l_1} \otimes I_n) & \cdots & \Phi_{NN}(\mathbf{1}_{l_N} \otimes I_n) \end{bmatrix}$$

and

$$\bar{D}e^P = \begin{bmatrix} \mathbf{1}_{l_1} \otimes \Psi_{11} & \cdots & \mathbf{1}_{l_1} \otimes \Psi_{1N} \\ \vdots & \ddots & \vdots \\ \mathbf{1}_{l_N} \otimes \Psi_{N1} & \cdots & \mathbf{1}_{l_N} \otimes \Psi_{NN} \end{bmatrix}$$

According to (3), for all integers $1 \leq i, j \leq N$,

$$\Phi_{ij}(\mathbf{1}_{l_j} \otimes I_n) = \mathbf{1}_{l_i} \otimes \Psi_{ij},$$

i.e. each block matrix Φ_{ij} has a constant row block-matrix sum Ψ_{ij}. This completes the first part of this lemma. In addition, it is straightforward to conclude that $\Psi_{ij} = Q_{ij}$ and $e^P = Q$. □

Plugging (2) into (1), one gets for $l \in \mathcal{I}$,

$$\dot{x}_l(t) = Ax_l(t) + BK(x_{j_l}(t) - x_l(t)) + BKh_{i_l}. \tag{4}$$

Constellate the states of these L agents in $x(t) := [x_1^T(t), x_2^T(t), \ldots, x_L^T(t)]^T$, such that for N integers $l_1, l_2, \cdots, l_N$ satisfying $\sum_{i=1}^N l_i = L$, the first l_1 agents have the same choice, the next l_2 agents have another common choice, and so on. The state equations in (4) can be put in a compact form:

$$\dot{x}(t) = (I_L \otimes A - \mathcal{L}_{\bar{\mathcal{G}}} \otimes (BK))x(t) + (I_L \otimes (BK))\bar{h}, \tag{5}$$

where $\bar{h} = [(\mathbf{1}_{l_1} \otimes h_{i_1})^T, (\mathbf{1}_{l_2} \otimes h_{i_2})^T, \cdots, (\mathbf{1}_{l_N} \otimes h_{i_N})^T]^T$, and the Laplacian of the graph $\bar{\mathcal{G}}$ takes the following form:

$$\mathcal{L}_{\bar{\mathcal{G}}} = \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1N} \\ L_{21} & L_{22} & \cdots & L_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ L_{N1} & L_{N2} & \cdots & L_{NN} \end{bmatrix} \tag{6}$$

where $L_{ij} \in \mathbb{R}^{l_i\times l_j}$, $i, j = 1, \cdots, N$.

Theorem 1. *Under assumption 1 and 2, for any initial condition $x(0)$, the multi-agent system (1) can achieve choice-based cluster consensus by using control protocol (2), if the block matrices L_{ij},$i, j = 1, \cdots, N$ of $\mathcal{L}_{\bar{\mathcal{G}}}$ have constant row sums.*

Proof. Solving equation (5), one obtains

$$x(t) = e^{\bar{A}t}x(0) + \int_0^t e^{\bar{A}(t-s)}(I_L \otimes (BK))\bar{h}ds, \quad (7)$$

where $\bar{A} = I_L \otimes A - \mathcal{L}_{\bar{\mathcal{G}}} \otimes (BK)$. Assume the eigenvalues of $\mathcal{L}_{\bar{\mathcal{G}}}$ are $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_L$. Then, we know from Lemma 1 that $\lambda_1 = 0$ and $Re(\lambda_i) > 0$ for $i = 2, 3, \cdots, L$. Thus, there exists an invertible matrix T which takes the form

$$T = [\mathbf{1}_L \ \ Y] \quad T^{-1} = \begin{bmatrix} r^T \\ Z \end{bmatrix} \quad (8)$$

where $r = [r_1, r_2, \cdots, r_L]^T$ is defined in Lemma 1, $Y \in \mathbb{R}^{L\times(L-1)}$ and $Z \in \mathbb{R}^{(L-1)\times L}$, transforms $\mathcal{L}_{\bar{\mathcal{G}}}$ to a Jordan form, i.e.,

$$T^{-1}\mathcal{L}_{\bar{\mathcal{G}}}T = J = diag\{0, \Delta\} \quad (9)$$

where $\Delta \in \mathbb{R}^{(L-1)\times(L-1)}$ is upper triangular. Therefore,

$$\begin{aligned} e^{\bar{A}t} =& (T \otimes I_n)e^{(I_L\otimes A - J\otimes(BK))t}(T^{-1} \otimes I_n) \\ =& (T \otimes I_n)\begin{bmatrix} e^{At} & 0 \\ 0 & e^{(I_{L-1}\otimes A - \Delta\otimes(BK))t} \end{bmatrix}(T^{-1} \otimes I_n). \end{aligned} \quad (10)$$

Since (A, B) is stabilizable, there exists a K such that $A - \lambda_i BK$ for all $i = 2, \cdots, L$ are Hurwitz as shown in [28, 29]. Thus, for such a K, we have $e^{\bar{A}t} \to (\mathbf{1}_L r^T) \otimes e^{At}$ as $t \to \infty$. It follows that

$$e^{\bar{A}t}x(0) \to ((\mathbf{1}_L r^T) \otimes e^{At})x(0), \text{ as } t \to \infty. \quad (11)$$

Write $\bar{A}$ in the same block matrix form as $\bar{M}$ in Lemma 2. Then $\bar{M}_{ij}$ have constant row block-matrix sums, since L_{ij} in (6) have constant row sums. Therefore, Lemma 2 indicates that the block matrices of the matrix $\int_0^t e^{\bar{A}(t-s)}ds$ partitioned in the same way as $\bar{A}$, also have constant row block-matrix sums. Concretely, $\int_0^t e^{\bar{A}(t-s)}ds$ can be rewritten as follows:

$$\begin{aligned} &\int_0^t e^{\bar{A}(t-s)}ds \\ =& (T \otimes I_n)\begin{bmatrix} \int_0^t e^{A(t-s)}ds & 0 \\ 0 & \int_0^t e^{(I_{L-1}\otimes A - \Delta\otimes(BK))(t-s)}ds \end{bmatrix} \\ &(T^{-1} \otimes I_n) \\ =& (\mathbf{1}_L r^T) \otimes \int_0^t e^{A(t-s)}ds + \bar{W}(t) \end{aligned} \quad (12)$$

where

$$\bar{W}(t) = \begin{bmatrix} \bar{W}_{11}(t) & \bar{W}_{12}(t) & \cdots & \bar{W}_{1N}(t) \\ \bar{W}_{21}(t) & \bar{W}_{22}(t) & \cdots & \bar{W}_{2N}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \bar{W}_{N1}(t) & \bar{W}_{N2}(t) & \cdots & \bar{W}_{NN}(t) \end{bmatrix} \quad (13)$$

is a time-varying $nL \times nL$ matrix and $\bar{W}_{ij}(t) \in \mathbb{R}^{nl_i \times nl_j}$ are time-varying block matrices having constant row block-matrix sums $\Theta_{ij}(t)$. If $A - \lambda_i BK$, $i = 2, \cdots, L$ can be made Hurwitz, then $\lim_{t\to\infty}\int_0^t e^{(I_{L-1}\otimes A - \Delta\otimes(BK))(t-s)}ds$ exists and is a constant matrix. It follows that $\bar{W}(t)$, $\bar{W}_{ij}(t)$ and $\Theta_{ij}(t)$, $1 \leq i, j \leq N$ also tend to constant matrices denoted by $\bar{W}$, $\bar{W}_{ij}$ and Θ_{ij}, respectively, as $t \to \infty$. Therefore,

$$\begin{aligned} &\int_0^t e^{\bar{A}(t-s)}ds(I_L \otimes (BK))\bar{h} \\ \to& [(\mathbf{1}_L r^T) \otimes \int_0^t e^{A(t-s)}ds + \bar{W}](I_L \otimes (BK))\bar{h} \\ =& ((\mathbf{1}_L r^T) \otimes \int_0^t e^{A(t-s)}ds)(I_L \otimes (BK))\bar{h} \\ &+ \begin{bmatrix} \sum_{j=1}^N \bar{W}_{1j}(\mathbf{1}_{l_j} \otimes (BKh_{i_j})) \\ \sum_{j=1}^N \bar{W}_{2j}(\mathbf{1}_{l_j} \otimes (BKh_{i_j})) \\ \vdots \\ \sum_{j=1}^N \bar{W}_{Nj}(\mathbf{1}_{l_j} \otimes (BKh_{i_j})) \end{bmatrix} \end{aligned} \quad (14)$$

as $t \to \infty$.

Combining (7), (11) and (14), one can derive that

$$\begin{aligned} x_l(t) \to& (r^T \otimes e^{At})x(0) \\ &+ \sum_{j=1}^N (\sum_{p=l_{j-1}+1}^{\sum_{q=1}^j l_q} r_p) \int_0^t e^{A(t-s)}ds(BK)h_{i_j} \\ &+ \sum_{j=1}^N \Theta_{lj}BKh_{i_j} \end{aligned} \quad (15)$$

as $t \to \infty$, where $\Theta_{lj} = \Theta_{1j}$ for $l = 1, \cdots, l_1$, $\Theta_{lj} = \Theta_{2j}$ for $l = l_1 + 1, \cdots, l_1 + l_2$, and so on. Note that the first two terms on the right hand side are identical for all $x_l(t)$, $l = 1, \ldots, L$. Thus, $x_l(t) - x_j(t) \to 0$ if $i_l = i_j$, i.e, agents l and j make the same choice. Also, for properly designed h_{i_l}, one can have $||x_l(t) - x_j(t)|| > 0$, as $t \to \infty$ if any two agents l and j make different choices. That is equivalent to say N-cluster consensus is achieved. This completes the proof. □

Remark 2. *The control protocol (2) can handle both stable and unstable identical agent dynamics A, while [17] only considered scalar systems with stable agent dynamics, and [18] used pinning control to drive agents with unstable dynamics to achieve cluster consensus, which does not work for stable system matrices. Also, From the proof of Theorem 1, one can find that differences between cluster consensus values are bounded, while in [18] and flocking problems [11, 13, 14], distances between clusters can go to infinity. [15] is an earlier work for agents modeled by first order integrators and there is no definition for distances between clusters.*

3.2 Clustering in unicyclic graphs

Theorem 1 says that the control protocol (2) can help to reach cluster consensus if L_{ij} have constant row sums, but it does not indicate the existence of such structures of the Laplacian $\mathcal{L}_{\bar{\mathcal{G}}}$. A trivial case is that all L agents in $\bar{\mathcal{G}}$ make the same choice, then $\mathcal{L}_{\bar{\mathcal{G}}}$ has a trivial partition itself and complete consensus of the whole group is achieved. For the unicyclic graph $\bar{\mathcal{G}}$, we give in this subsection more nontrivial and interesting cases satisfying this structural condition.

According to the definition of directed trees, it is hard to tell which node is the root of the attached trees in unicyclic graphs, since every node has a parent. In this paper, we call the node which is contained in $\mathcal{C}$ and has directed trees attached to it be the root of its attached trees. For example, in Figure 1, node 3 is the root of the tree consists of nodes $\{3,5\}$ and node 4 is the root of the tree consists of nodes $\{4,5,6\}$. Assume that the depth of the longest tree is d. Let all agents forming the cycle (for example, agents 1-4 in Figure 1) be at the first level, the highest level; all children of the first level agents (e.g. agents 5 and 6 in Figure 1) be at a lower level, the second level, and so on. Then, there are totally d levels.

If all of the agents in the same level make a common choice, and the choices of different levels are distinct to each other, then, one can verify that all L_{ij} in the resulting Laplacian (6) have constant row sums. More generally, all agents at consecutive levels starting from the first level can be grouped together to have a common choice, while for the remaining lower levels an exclusive choice should be owned at each level.

Corollary 1. *Under assumption 1 and 2, if all agents at consecutive levels starting from the first level have a common choice, while for the remaining lower levels an exclusive choice is made at each level, then, for any initial condition* $x(0)$*, the multi-agent system (1) can achieve choice-based cluster consensus by using control protocol (2).*

4 Simulation

In this section, we provide simulation examples to illustrate our results. The topology of the graph to be used is given in Fig. 1. Its Laplacian matrix is given by:

$$\mathcal{L}_{\bar{\mathcal{G}}} = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdot & 0 & 0 & \cdot & 0 & 0 \\ 0 & 1 & -1 & 0 & \cdot & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdot & 0 & 0 & \cdot & 0 & 0 \\ -1 & 0 & 0 & 1 & \cdot & 0 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & -1 & 0 & \cdot & 1 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & -1 & \cdot & 0 & 1 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & 0 & -1 & \cdot & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & -1 & \cdot & 0 & 1 \end{bmatrix} \tag{16}$$

which is partitioned to 9 block matrices. It is easily seen that these block matrices have constant row sums. Also, one can partition from the first 6 rows and columns to construct a 2-by-2 block matrix. The spectrum of $\mathcal{L}_{\bar{\mathcal{G}}}$ is $\{0,1,1\pm j,2\}$ where $j=\sqrt{-1}$, and Lemma 1 is verified.

The agent dynamics is described by a second order model:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{17}$$

which is stabilizable.Since the first coordinate is stable and uncontrollable, so, we only need to consider the second coordinate. The controller gain is chosen as $K = [0,\ 2]$ such that $A - \lambda_i BK$ is stable for all $\lambda_i, i = 2,\cdots,8$. $h_1 = [0,-2]^T, h_2 = [0,0]^T, h_3 = [0,2]^T$. Simulation results for the second coordinates of $x_{1l}(t) := x_1(t) - x_l(t)$ for all $l \in \mathcal{I}$ are exhibited for the following different cases.

(1) Assume that agents at the first level choose 1, i.e, $i_l = 1$, for $l = 1,\cdots,4$, agents of the second level choose 2, i.e, $i_l = 2$, for $l = 5,6$ and agents at the third level choose 3, i.e, $i_l = 3$, for $l = 7,8$. Fig.2 shows that choice-based 3-cluster consensus is achieved.

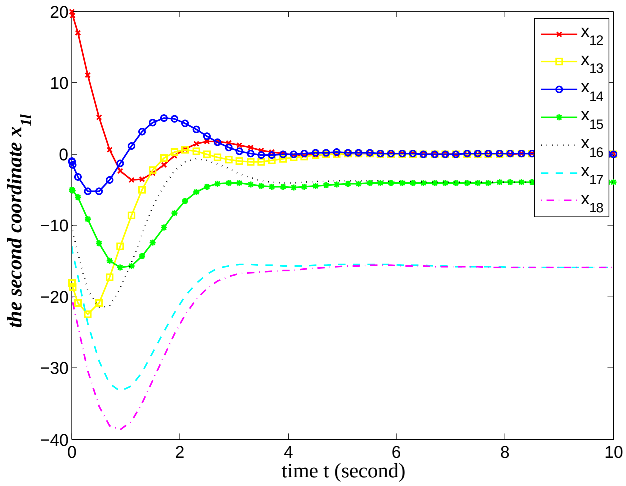

Fig. 2: Choice-based 3-cluster consensus.

(2) Assume that agents at the first level and the second level choose 1, i.e, $i_l = 1, l = 1,\cdots,6$, and agents at the third level choose 2, i.e, $i_l = 2, l = 7,8$. Fig.3 shows that choice-based 2-cluster consensus is achieved.

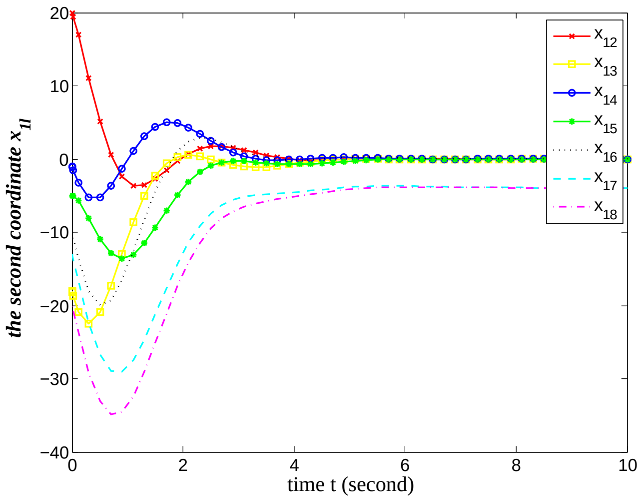

Fig. 3: Choice-based 2-cluster consensus.

(3) Assume that agents of the first level and the third level choose 1, i.e, $i_l = 1, l = 1,\cdots,4,7,8$, and agents at the second level choose 2, i.e, $i_l = 2, l = 5,6$. So the grouping rules in Corollary 1 is violated. Fig.4 shows that the whole group splits into three clusters but only two choices are made by the agents. So it is a cluster consensus in the sense of [15, 17], but not a choice-based cluster consensus as claimed in Remark 1.

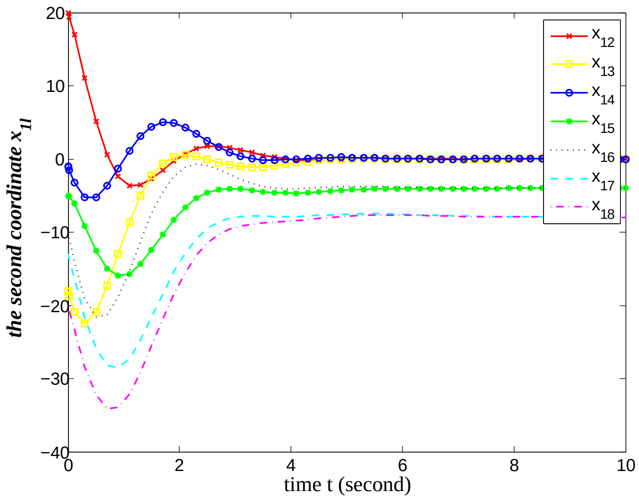

Fig. 4: Two choices result in three clusters

5 Conclusion

This paper studied choice-based actions in a unicyclic multi-agent network, where each agent has only local information. Choice-based cluster consensus is investigated for multiple agents with identical generic linear models. A distributed control protocol is proposed such that agents make the same choices can reach a common consensus state value, while agents make different choices will reach distinct state values. Especially, a grouping method for unicyclic graphs is presented such that the structural condition is satisfied. For future studies, one can consider dynamically changing graph topologies, so that agents can change their neighbors to satisfy the structural condition.

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