Power Control for Non-Gaussian Interference

Yi Chen and Wing Shing Wong, *Fellow, IEEE*

***Abstract*—This paper investigates a wireless communication system where the mutual user interference is not assumed to be a Gaussian process. We derive an exact expression for the average bit error probability (BEP) for such a system and study the non-Gaussian interference model through two types of power control problems. We analyze the situation under which the system can be asymptotically error-free, the behavior of users' BEP when scaling up a fixed power setting by a uniform scalar and the effect of varying symbol rate on the system performance. Our work shows that the non-Gaussian model has significantly different performance characteristics from the traditional Gaussian interference model. Simulations also show that the Gaussian model is generally pessimistic in comparison with the non-Gaussian model.**

***Index Terms*—Non-Gaussian model, power control, bit error probability (BEP), character matrix, M-matrix.**

I. Introduction

IN modern wireless communication systems, mutual user interference is one of the fundamental factors that limit performance. Enormous amount of research has been devoted to use power control to manage the interference (ref. [2]–[5] and the references therein). Most of those papers start with the assumption that the signal-to-interference-plus-noise power ratio (SINR) is the utility metric. SINR has clearly understood implication on bit error probability (BEP), capacity and other Quality of Service (QoS) metrics mainly for additive Gaussian channels [6]. In other words, using SINR as a surrogate for QoS metrics such as BEP or capacity implicitly assumes that for any given user, the combined interference from other users sharing the same spectrum is a Gaussian process. This Gaussian approximation can be partially justified by the Central Limit Theorem if the interference comes from mutually independent, identically distributed user signal processes and if the number of such users is large (the property of identically distributed is required unless certain conditions are satisfied [7]). However, in practice, the number of interfering users may be small or the interfering signals may not be independent or identically distributed. When the Central Limit Theorem is not applicable, the Gaussian model may not accurately capture system performance and the significance of SINR as a utility metric is less clear. In [8]–[14], the statistic distribution of interference and the BEP expression were studied for different wireless networks, fading channels and spacial distribution of the interfering users. It is shown that the Gaussian model can yield poor BEP estimates under certain operating conditions.

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The authors are with the Department of Information Engineering, The Chinese University of Hong Kong, N.T., Hong Kong (e-mail: {cy007, wswong}@ie.cuhk.edu.hk).

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In this paper, we study a system model where the interference is not assumed to be Gaussian. We call it the *non-Gaussian model*. Different from [8]–[14], our model allows for different symbol rates and powers for different users and we focus on studying the non-Gaussian model for power control. We derive an exact expression of BEP for such a system and use BEP as the QoS metric to investigate two types of power control problems under the new model. The first problem aims at minimizing the maximal BEP of all users. We call it the *minimal BEP problem*. The second problem aims at minimizing the total transmission power while satisfying the BEP requirement of each user. It is referred to as the *minimal power problem*. Under the traditional Gaussian model, since the BEP is a monotonically decreasing function of the SINR, the two problems are equivalent to the power balancing problem [2] and the SINR tracking problem [3] respectively. However, a major finding of our work is that the non-Gaussian model has some significantly different performance characteristics from the Gaussian model. They are reflected in the following three major aspects.

The first concerns the feasible BEP region, that is the set of BEP vectors of all users that can be achieved through power control. Under the Gaussian model, it is a well-known result of the power balancing problem that the maximum of the minimal SINR of all users is related to the dominant eigenvalue of the channel gain matrix [2] and therefore is bounded. Hence, it is not possible to make the BEP of all users arbitrarily small. While for the non-Gaussian model, it is possible that the BEP of all users approach simultaneously to zero under certain technical condition which is explicitly characterized here.

The second performance difference shows up when we scale up a fixed power setting by a uniform multiplier. Given an initial power setting, the SINR of all users increase monotonically with the multiplier. Under the Gaussian model, this implies the solution to the minimal BEP problem is obtained by letting the power vector, or equivalently the multiplier, approach infinity. However, under the non-Gaussian model, the behavior of BEP with respect to the multiplier is more complicated. We show that for certain channel gain matrices, the BEP for some users first decrease monotonically then increase monotonically with the multiplier. It is shown by example that the solution to the minimal BEP problem can be achieved by a finite power vector under the non-Gaussian model.

The third performance difference occurs in asynchronous transmission systems where different users have different symbol rates. In many papers reported in the literature, the variation in interference power with respect to the symbol rate and asynchronism is not explicitly considered. That is, the variances of the interference are typically assumed to be independent of the symbol rate. So varying the symbol rate of one user does not change the BEP of other users. The non-Gaussian model, however, shows otherwise. For example, under the assumption of a fixed power setting, if we decrease the symbol rate of one user, the BEP of other users will increase.

In this paper we demonstrated by rigorous arguments the qualitative differences between the Gaussian and non-Gaussian models. We also performed simulations to numerically compare the two models for coded and uncoded channels. It is shown that the Gaussian model is generally pessimistic for both cases. From the results reported in this paper, one can expect that the interference can be ameliorated under the non-Gaussian model by appropriately choosing the symbol rate, exploiting the interference structure and so on, whereas the Gaussian model entails an a priori loss of this possibility. Our long term aim is to investigate modulation/demodulation schemes that can efficiently use the structure of the interference to mitigate its effect and to optimize the network throughput by means of better resource allocation algorithms. This is obviously an ardor and complicated mission. In this paper, we focus on clarifying some basic properties of the non-Gaussian model.

The rest of the paper is organized as follows. The system model and the error probability calculation are presented in Section II. Section III analyzes the minimal BEP problem and proves the necessary and sufficient condition for a system to be asymptotically error-free. In Section IV, the behavior of BEP with respect to the multiplier is described for certain channel gain matrices. Some bounds of the minimal BEP are also given. Section V investigates the minimal power problem and introduces some properties of the BEP function. Simulation results are provided in Section VI. Finally, in Section VII, we give some concluding remarks.

II. System Model and Error Probability Calculation

Consider a general wireless communication system with n transmitters $\{\mathrm{tran}_i : i = 1, \ldots, n\}$ and n receivers $\{\mathrm{rec}_i : i = 1, \ldots, n\}$, in which, tran_i communicates to rec_i and all the transmissions share the same wireless radio spectrum. We refer user i to be the pair $(\mathrm{tran}_i, \mathrm{rec}_i)$. Let x_i^2 be the transmitted power of tran_i where $x_i > 0$ is the amplitude of the transmitted signal. Define $\mathbf{x} = (x_1, \ldots, x_n)$. Assume slow and flat fading. Let h_{ij}^2 be the power gain between tran_j and rec_i where $h_{ij} > 0$ is the amplitude attenuation factor on x_j. There is no assumption on the statistic distribution of h_{ij}. The channel gain matrix is $\mathbf{H} = (h_{ij})$. We consider a snapshot of the system, and thus h_{ij} is treated as a constant. Its magnitude reflects the effect of path loss, shadow fading and antenna gains.

All transmitters apply binary phase-shift keying (BPSK) modulation. The analysis in this paper can be extended to QPSK. Let R_i be the symbol/bit rate of tran_i and p_{T_i} be the unit-amplitude rectangular pulse of duration $T_i = 1/R_i$. Let $\{b_i^k\}_{k=-\infty}^{\infty}$ denote the BPSK-modulated information sequence of tran_i where b_i^k is uniformly distributed on $\mathbb{B} = \{+1, -1\}$. Assume there is no frequency offset and phase offset in all the transmitters and receivers. Thus the carrier is suppressed for notational economy. The baseband signal $s_i(t)$ of tran_i is

$$s_i(t) = x_i a_i(t), \tag{1}$$

where

$$a_i(t) = \sum_{k=-\infty}^{\infty} b_i^k p_{T_i}(t - kT_i). \tag{2}$$

The transmitted signals from all the transmitters may not necessarily be synchronized, unless explicitly stated otherwise. At rec_i, the received baseband signal $r_i(t)$ is

$$r_i(t) = \sum_{j=1}^{n} h_{ij} x_j a_j(t - \tau_{ij}) + n_i(t), \tag{3}$$

where τ_{ij} is the time delay of $s_j(t)$ at rec_i and $n_i(t)$ is the additive white Gaussian noise (AWGN) with two-sided power spectral density $N_0/2$.

A receiver demodulates the received baseband signal using a matched filter, followed by a threshold decision. The impulse response $s_i^0(t)$ of the filter of rec_i is a rectangular pulse of amplitude 1 and duration T_i. Without loss of generality, assume $\tau_{ii} = 0$, i.e., the matched filter of rec_i is synchronized to the arrival signal transmitted by tran_i. Consider a bit interval as $[0, T_i]$ to be demodulated. Label the first bit overlapping with this interval by b_j^0 for all j and the sequential bits are $b_j^1, b_j^2, \ldots$. Without confusion, we use τ_{ij} to represent the misalignment of the interfering symbol of user j with respect to the desired symbol of user i, and it is assumed to be uniformly distributed in $[0, T_j)$ (see Fig. 1). The input to the decision device for rec_i is

$$y_i = \int_0^{T_i} r_i(t) s_i^0(t) \mathrm{d}t = W_{ii} + \sum_{j \neq i} W_{ij} + Z_i, \tag{4}$$

where

$$W_{ij} = h_{ij} x_j \int_0^{T_i} a_j(t - \tau_{ij}) \mathrm{d}t, \quad Z_i = \int_0^{T_i} n_i(t) s_i^0(t) \mathrm{d}t. \tag{5}$$

Since $\tau_{ii} = 0$, $W_{ii} = b_i^0 T_i h_{ii} x_i$. Unlike classical models, the interference term $\sum_{j \neq i} W_{ij}$ is not assumed to be a Gaussian random variable. For all $j \neq i$, W_{ij} is a random variable depending on τ_{ij} and the information bits b_j^k of tran_j. Several typical cases of the integral $\int_0^{T_i} a_j(t - \tau_{ij}) \mathrm{d}t$ are illustrated in Fig. 1. For example, when the interfering signal and the intended signal are synchronous (Fig. 1(a)), $W_{ij} = h_{ij} x_j T_i b_j^0$; when they are asynchronous and $T_i > T_j$ (Fig. 1(c)), $W_{ij} = h_{ij} x_j (\tau_{ij} b_j^0 + T_j (b_j^1 + b_j^2 + b_j^3) + (T_i - \tau_{ij} - 3T_j) b_j^4)$.

For the threshold decision, an error occurs if $y_i < 0$ when $b_i^0 = 1$, or if $y_i > 0$ when $b_i^0 = -1$. Since b_i^0 takes ± 1 with equal probability, the average BEP is equal to the probability of receiving $y_i < 0$ when $b_i^0 = 1$. Since Z_i is a Gaussian random variable with zero mean and variance $\sigma_i^2 = N_0 T_i / 2$, the BEP of user i conditioned on $\sum_{j \neq i} W_{ij}$ is

$$\Pr\left(y_i < 0 | \sum_{j \neq i} W_{ij}, b_i^0 = 1 \right) = Q\left(\frac{T_i h_{ii} x_i + \sum_{j \neq i} W_{ij}}{\sigma_i} \right) \tag{6}$$

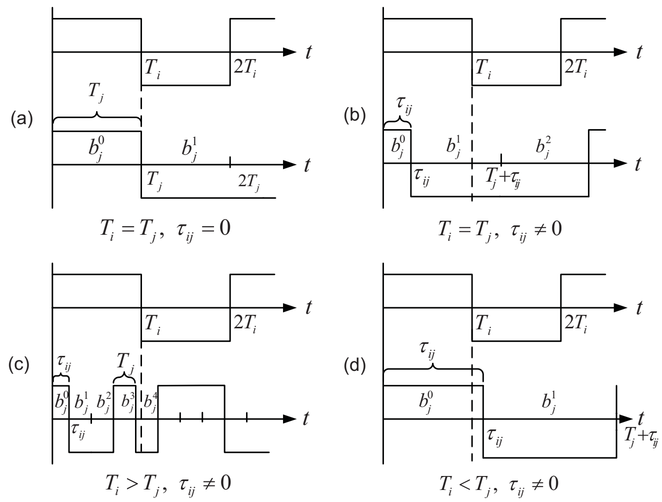

Fig. 1. Typical cases of interfering signal with symbol duration T_j in the integration interval $[0, T_i]$. τ_{ij} is the relative time offset and b_j^k is the information bit.

where $Q(\cdot)$ is the complementary error function defined as $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{u^2}{2}} \mathrm{d}u$.

Hence, the average BEP is

$$\lambda_i(\mathbf{x}) = \mathrm{E}[\Pr(y_i < 0 | \sum_{j \neq i} W_{ij}, b_i^0 = 1)], \tag{7}$$

where the expectation is over $\{b_j^k : j \neq i\}$ and $\{\tau_{ij} : j \neq i\}$.

In the following, we give a review of the Gaussian model. Under the Gaussian model, the output interference from the matched filter, $\sum_{j \neq i} W_{ij}$, given by (5), is approximated by a Gaussian random variable with identical variance. So the decision statistic y_i, given by (4), is a Gaussian random variable. This yields the BEP

$$\lambda_i^G(\mathbf{x}) = \Pr\left(y_i < 0 | b_i^0 = 1\right) = Q\left(\frac{W_{ii}}{\sqrt{\sum_{j \neq i} \sigma_{W_{ij}}^2 + \sigma_i^2}}\right), \tag{8}$$

where $\sigma_{W_{ij}}^2$ is the variances of W_{ij}.

In the scenario of asynchronous transmission ($\tau_{ij} \neq 0$) and different bit rates ($R_i \neq R_j$) (see Fig. 1(b)(c)(d)), averaging over the distribution of $\{b_j^k : j \neq i\}$ and $\{\tau_{ij} : j \neq i\}$ yields (the derivation is given in [15]; the case for $R_j = R_i$ while $\tau_{ij} \neq 0$ can be found in [8], [9]),

$$\begin{aligned} \sigma_{W_{ij}}^2 &= h_{ij}^2 x_j^2 \min\{T_i^2, T_j^2\} \left(\max\{1, \frac{T_i}{T_j}\} - \frac{1}{3}\min\{1, \frac{T_i}{T_j}\}\right) \\ &< h_{ij}^2 x_j^2 T_i^2. \end{aligned} \tag{9}$$

However, in many papers reported in the literature, it is implicitly assume that all users transmit at the same bit rate synchronously (see Fig. 1(a)) [5]. In that case the variance $\sigma_{W_{ij}}^2$ is equal to $h_{ij}^2 x_j^2 T_i^2$. We call this special case the Aligned Gaussian model and define the corresponding SINR by

$$\gamma_i(\mathbf{x}) = \frac{W_{ii}^2}{\sum_{j \neq i} \sigma_{W_{ij}}^2 + \sigma_i^2} = \frac{h_{ii}^2 x_i^2}{\sum_{i \neq j} h_{ij}^2 x_j^2 + \sigma_i^2 / T_i^2}. \tag{10}$$

The BEP under the Aligned Gaussian model becomes

$$\lambda_i^G(\mathbf{x}) = Q(\sqrt{\gamma_i(\mathbf{x})}). \tag{11}$$

Note that the Aligned Gaussian model yields a more pessimistic result since it assumes a larger interference variance among the two classes of Gaussian models (Aligned and Misaligned, see the last inequality in (9)). In subsequent discussion, the Aligned Gaussian model is considered while the simulation results for the Misaligned Gaussian model are also presented in Section VI.

In the following power control optimization problems, the channel gains are known and the optimization variables are the signal amplitudes.

III. The Minimal BEP Problem

In this section, we first recall the solution to the minimal BEP problem under the Gaussian model (or equivalently the power balancing problem). Then we analyze the situation under which the minimal BEP can approach zero under the non-Gaussian model. Scalar operators, such as " $>$", " $\geq$" or " $=$" are applied to vectors component-wise. We use "$|\cdot|$" to denote the absolute value of a scalar and "$\lceil\cdot\rceil$" to denote the ceiling function.

The power balancing problem under the Gaussian model is to maximize the minimal SINR of all users. It is equivalent to the minimal BEP problem and can be written as

$$\hat{\lambda}^G = \inf_{\mathbf{x} > 0} \max_{i=1,\ldots,n} \lambda_i^G(\mathbf{x}), \tag{12}$$

where $\lambda_i^G(\mathbf{x})$ is defined in (11). It can be solved by finding the Perron-Frobenius eigenvalue $\rho_{\mathbf{Z}}$ of the normalized power gain matrix $\mathbf{Z} = (z_{ij}) = (\frac{h_{ij}^2}{h_{ii}^2})$ and its corresponding eigenvector $[\hat{x}_1^2, \ldots, \hat{x}_n^2]^\top$, given that $\mathbf{Z}$ is an irreducible matrix [2]. Let $\mathbf{x}_{\mathbf{Z}} = (\hat{x}_1, \ldots, \hat{x}_n)$, then

$$\hat{\lambda}^G = \lim_{\alpha \to \infty} \max_{i=1,\ldots,n} \lambda_i^G(\alpha \mathbf{x}_{\mathbf{Z}}) = Q\left(\sqrt{\frac{1}{\rho_{\mathbf{Z}} - 1}}\right). \tag{13}$$

Since $\mathbf{Z}$ is a nonnegative matrix with diagonal entries equal to one, $\rho_{\mathbf{Z}} > 1$, and thus $\hat{\lambda}^G > 0$. That is, under the Gaussian model, for any given channel gains, it is not possible to make the BEP of all users approach zero. For the non-Gaussian model, it is possible that the BEP of all users go simultaneously to zero under certain technical condition which is explicitly characterized below. To make the presentation precise, we introduce the following definitions.

Definition 1. *An n-user wireless communication system is said to be asymptotically error-free if the minimal BEP*

$$\hat{\lambda} = \inf_{\mathbf{x} > 0} \max_{i=1,\ldots,n} \lambda_i(\mathbf{x}) = 0, \tag{14}$$

where $\lambda_i(\mathbf{x})$ is the BEP function.

Remark: This definition of asymptotically error-free transmission is focused on the decision error at the physical layer. It is different from the information-theoretic concept in coding theory.

Definition 2. *A matrix $\mathbf{A} = (a_{ij})$ is said to be row diagonally dominant if*

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \quad \textit{for all } i. \tag{15}$$

Definition 3. [16] *A square matrix is called a Z-matrix if all off-diagonal entries are less than or equal to zero. A Z-matrix $\mathbf{A}$ is called an M-matrix if it satisfies any one of the following equivalent conditions:*

1) *The eigenvalues of $\mathbf{A}$ all have positive real parts.*
2) $\mathbf{Av} \geq 0$ *implies* $\mathbf{v} \geq 0$.
3) *There exists a vector $\mathbf{v}$ with positive entries such that $A\mathbf{v} > 0$.*
4) *The diagonal entries of $\mathbf{A}$ are positive and $\mathbf{AD}$ is row diagonally dominant for some positive diagonal matrices $\mathbf{D}$.*

Definition 4. *The* character matrix *of an n-user wireless communication system with channel gain matrix $\mathbf{H} = (h_{ij})$ is an $n \times n$ matrix $\mathbf{C} = (c_{ij})$ with $c_{ii} = h_{ii}$ for all i and $c_{ij} = -h_{ij}$ for $j \neq i$.*

Obviously, $\mathbf{C}$ is a Z-matrix. One type of the M-matrix is row diagonally dominant matrix.

Theorem 1. *Under the non-Gaussian model, an n-user wireless communication system is asymptotically error-free if and only if its character matrix is an M-matrix.*

Proof: Consider an n-user wireless communication system with channel gain matrix $\mathbf{H} = (h_{ij})$ and character matrix $\mathbf{C}$. First fix a receiver i. From (5), we see that for $j \neq i$,

$$W_{ij} = h_{ij} x_j \int_0^{T_i} a_j(t - \tau_{ij}) \mathrm{d}t \geq -T_i h_{ij} x_j, \qquad (16)$$

where the last equality is obtained when $a_j(t - \tau_{ij}) = -1$ for $0 \leq t \leq T_i$. That is, the interfering bits b_j^k for $0 \leq k \leq \lceil T_i/T_j \rceil$ that fall onto the period $[0, T_i]$ are -1. For example, in Fig. 1(b), $W_{ij} = h_{ij} x_j (\tau_{ij} b_j^0 + (T_i - \tau_{ij}) b_j^1) \geq -h_{ij} x_j T_i$ for any τ_{ij} and b_j^0, b_j^1. Since $Q(\cdot)$ is a decreasing function, by (7) and (16),

$$\lambda_i(\mathbf{x}) = \mathrm{E}\left[Q\left(\frac{W_{ii} + \sum_{j \neq i} W_{ij}}{\sigma_i}\right)\right] \qquad (17)$$

$$\leq \mathrm{E}\left[Q\left(\frac{T_i h_{ii} x_i - \sum_{j \neq i} T_i h_{ij} x_j}{\sigma_i}\right)\right] \qquad (18)$$

$$= Q\left(\frac{\tilde{y}_i T_i}{\sigma_i}\right), \qquad (19)$$

where the expectation is over $\{b_j^k : j \neq i\}$ and $\{\tau_{ij} : j \neq i\}$; $\tilde{y}_i = h_{ii} x_i - \sum_{j \neq i} h_{ij} x_j$ is defined for the worst interfering bits pattern, i.e., b_j^k falling on $[0, T_i]$ are -1 for all $j \neq i$. Let $\tilde{\mathbf{y}} = [\tilde{y}_1, \ldots, \tilde{y}_n]^\top$. It follows from Definition 4 that $\tilde{\mathbf{y}} = \mathbf{Cx}$. If $\mathbf{C}$ is an M-matrix, by the third equivalent condition of M-matrix, there exists $\mathbf{x}^* > 0$ such that $\tilde{\mathbf{y}}^* = \mathbf{Cx}^* > 0$. Let $\{\alpha \mathbf{x}^* : \alpha > 0\}$ be a sequence of input signal amplitude with $\alpha \to \infty$. Then, $\lambda_i(\alpha \mathbf{x}^*) \leq Q(\alpha \tilde{y}_i^* T_i / \sigma_i) \to 0$ as $\alpha \to \infty$ for all i. Hence $\hat{\lambda} = 0$, that is, the system is asymptotically error-free.

On the other hand, if $\mathbf{C}$ is not an M-matrix, given any $\mathbf{x} > 0$, there exists at least one entry in $\mathbf{Cx}$, say $\tilde{y}_i$, which is not positive. Let

$$P_i = \Pr\{b_j^k = -1 : j \neq i, 0 \leq k \leq \lceil T_i/T_j \rceil\} \qquad (20)$$
$$= 2^{-\sum_{j \neq i}(\lceil T_i/T_j \rceil + 1)}.$$

Then,

$$\lambda_i(\mathbf{x}) \geq P_i Q(\tilde{y}_i T_i / \sigma_i) \geq P_i Q(0) = P_i/2 > 0. \qquad (21)$$

This means that $\hat{\lambda} > 0$ and the system is not asymptotically error-free. ■

Under the Gaussian model, the minimal BEP $\hat{\lambda}^G = \lim_{\alpha \to \infty} \lambda_i^G(\alpha \mathbf{x_Z}) > 0$ (refer to (13)) and thus the system will never be asymptotically error-free. In contrast, under the non-Gaussian model, if the character matrix $\mathbf{C}$ is an M-matrix, $\hat{\lambda} = \lim_{\alpha \to \infty} \lambda_i(\alpha \mathbf{x}^*) = 0$ (refer to the proof of Theorem 1) and the system is asymptotically error-free. What will happen if $\mathbf{C}$ is not an M-matrix? Note that for both models if $\mathbf{C}$ is an M-matrix, the minimal BEP is approached by letting α go to infinity. This is due to the fact that given $\mathbf{x_Z}$ and $\mathbf{x}^*$, $\lambda_i^G(\alpha \mathbf{x_Z})$ and $\lambda_i(\alpha \mathbf{x}^*)$ are monotonically decreasing with α for all i. However, for the non-Gaussian model with $\mathbf{C}$ being not an M-matrix, given any $\mathbf{x}$, the monotonic property of $\lambda_i(\alpha \mathbf{x})$ with respect to α may not hold for some i. As a result, solving the minimal BEP problem for this case is quite complicated. In subsequent section we study this problem in a synchronized transmission scenario.

IV. Non-Gaussian Model with a Non-M-Matrix Character Matrix

In this section, we study the minimal BEP problem under the non-Gaussian model when the character matrix is not an M-matrix. Assume all users transmit at the same bit rate $R = 1/T$ and the transmitted signals are synchronized (as illustrated in Fig. 1(a)). Then the interference statistic, W_{ij}, given by (5), is $W_{ij} = b_j^0 h_{ij} x_j T$. For the notation simplicity, b_j is used instead of b_j^0 in this section. Assume $\sigma_i^2/T = 1$. By (7), the BEP of user i is

$$\lambda_i(\mathbf{x}) = \frac{1}{2^{n-1}} \sum_{\substack{b_{j'} \in \mathbb{B} \\ j' \neq i}} Q\left(h_{ii} x_i + \sum_{j \neq i} b_j h_{ij} x_j\right). \qquad (22)$$

Define a mapping $\phi : \mathbb{B}^{n-1} \to \mathbb{Z}$ as $\phi(a_1, a_2, \ldots, a_{n-1}) = \sum_{i=1}^{n-1} \frac{a_i+1}{2} 2^{i-1}$. It is a notional device. We can check that ϕ is a one-to-one mapping and $\phi[\mathbb{B}^{n-1}] = \{0, 1, \ldots, 2^{n-1}-1\}$. For integer $0 \leq k \leq 2^{n-1} - 1$, define $y_i^k(\mathbf{x}) = h_{ii} x_i + \sum_{j \neq i} b_j h_{ij} x_j$, where $(b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n) = \phi^{-1}(k)$. Comparing with (22), the BEP can be alternatively written as

$$\lambda_i(\mathbf{x}) = \frac{1}{2^{n-1}} \sum_{k=0}^{2^{n-1}-1} Q\left(y_i^k(\mathbf{x})\right). \qquad (23)$$

Note that $y_i^0(\mathbf{x}) = h_{ii} x_i - \sum_{j \neq i} h_{ij} x_j$. So for $k > 0$, $y_i^k(\mathbf{x}) > y_i^0(\mathbf{x})$. Let $\mathbf{y}^0(\mathbf{x}) = (y_i^0(\mathbf{x}) : i = 1, \ldots, n)$. We see $\mathbf{y}^0(\mathbf{x}) = \mathbf{Cx}$.

Lemma 1. *If a character matrix $\mathbf{C}$ is not an M-matrix, then for any $\mathbf{x} > \mathbf{0}$ either at least one entry in $\mathbf{Cx}$ is negative or $\mathbf{Cx} = \mathbf{0}$.*

Proof: We prove this by contradiction. Assume that there exists $\mathbf{x} > \mathbf{0}$ such that all the entries in $\mathbf{y}^0(\mathbf{x}) = \mathbf{Cx}$ are nonnegative and $\mathbf{y}^0(\mathbf{x}) \neq \mathbf{0}$. Let $\Phi = \{i : y_i^0(\mathbf{x}) = 0\}$. If $\Phi = \emptyset$, it contradicts with the condition that $\mathbf{C}$ is not an M-matrix. Since $\mathbf{y}^0(\mathbf{x}) \neq \mathbf{0}$, there exists $i \notin \Phi$ such that $y_i^0(\mathbf{x}) >$

0. Find $0 < \epsilon < y_i^0(\mathbf{x})/h_{ii}$. Let $\mathbf{x}' = (x_1, \ldots, x_i - \epsilon, \ldots, x_n)$. We have $\mathbf{x}' > 0$ and $\mathbf{C}\mathbf{x}' > \mathbf{0}$, i.e., a contradiction to the fact that $\mathbf{C}$ is not an M-matrix. ■

Definition 5. *Define sets*

1) $\mathcal{X}_{\mathbf{C}} = \{\mathbf{x} > \mathbf{0} : \mathbf{C}\mathbf{x} > \mathbf{0}\}$
2) $\mathcal{X}_0 = \{\mathbf{x} > \mathbf{0} : \mathbf{C}\mathbf{x} = \mathbf{0}\}$
3) $\mathcal{X}_1 = \{\mathbf{x} > \mathbf{0} : z_{i,l}(\mathbf{x}) = h_{ii}x_i - \sum_{j\neq i,l} h_{ij}x_j > 0, i = 1, \ldots, n, l \neq i\}$.

Note that $z_{i,l}(\mathbf{x}) = y_i^0(\mathbf{x}) + h_{il}x_l$. Moreover, $\mathcal{X}_{\mathbf{C}}, \mathcal{X}_0$ and $\mathcal{X}_1$ are cones. $\mathcal{X}_{\mathbf{C}} \subset \mathcal{X}_1$ and $\mathcal{X}_0 \subset \mathcal{X}_1$.

Theorem 2. *Under the non-Gaussian model, for an n-user wireless communication system with synchronized transmission, if the character matrix is not an M-matrix, the minimal BEP $\hat{\lambda} \geq 1/2^n$. Furthermore, if there exists $\mathbf{x}_0 \in \mathcal{X}_0$, $\hat{\lambda} = \lim_{\alpha\to\infty} \max_{i=1,\ldots,n} \lambda_i(\alpha\mathbf{x}_0) = \frac{1}{2^n}$.*

Proof: If the character matrix $\mathbf{C}$ is not an M-matrix, for any $\mathbf{x} > \mathbf{0}$, there exists a user i such that $y_i^0(\mathbf{x}) \leq 0$ (see Lemma 1). So we have

$$\hat{\lambda} \geq \inf_{\mathbf{x}>0} \max_{i=1,\ldots,n} \frac{1}{2^{n-1}} Q\left(y_i^0(\mathbf{x})\right) \geq \frac{1}{2^{n-1}} \cdot \frac{1}{2} = \frac{1}{2^n}. \quad (24)$$

If there exists $\mathbf{x}_0 \in \mathcal{X}_0$, then for all i, $y_i^0(\mathbf{x}_0) = 0$ and $y_i^k(\mathbf{x}_0) > 0$ for $k > 0$. We have

$$\begin{aligned}\hat{\lambda} &\leq \lim_{\alpha\to\infty} \max_{i=1,\ldots,n} \lambda_i(\alpha\mathbf{x}_0)\\ &= \max_{i=1,\ldots,n} \frac{1}{2^{n-1}} \sum_{k=1}^{2^{n-1}-1} \lim_{\alpha\to\infty} Q(\alpha y_i^k(\mathbf{x}_0)) + \frac{1}{2^{n-1}} Q(0)\\ &= \frac{1}{2^{n-1}} \cdot \frac{1}{2} = \frac{1}{2^n}.\end{aligned}$$

Together with (24), $\hat{\lambda} = \lim_{\alpha\to\infty} \max_{i=1,\ldots,n} \lambda_i(\alpha\mathbf{x}_0) = \frac{1}{2^n}$. ■

When $\mathcal{X}_0 = \emptyset$, given any $\mathbf{x}$, there exists a user i such that $y_i^0(\mathbf{x}) < 0$ and thus $\lambda_i(\alpha\mathbf{x})$ may not be monotonically decreasing with α. In subsequent discussion, we consider a class of character matrix for which the monotonic property of $\lambda_i(\alpha\mathbf{x})$ with respect to α does not hold. An example is shown that the solution to the minimal BEP problem is achieved by a finite power vector.

Given any $\mathbf{x} > 0$, define $\mathcal{I}_-(\mathbf{x}) = \{i : y_i^0(\mathbf{x}) < 0\}$. $\overline{\mathcal{I}}_-(\mathbf{x})$ is the complement of $\mathcal{I}_-(\mathbf{x})$.

Lemma 2. *For any $\mathbf{x} > \mathbf{0}$ but $\mathbf{x} \notin \mathcal{X}_1$, $\mathcal{I}_-(\mathbf{x})$ is nonempty and there exists $i \in \mathcal{I}_-(\mathbf{x})$ such that $\lambda_i(\mathbf{x}) \geq \frac{1}{2^{n-1}}$.*

Proof: Since $\mathbf{x} \notin \mathcal{X}_1$, there exists i and l with $l \neq i$ such that $z_{i,l}(\mathbf{x}) \leq 0$. So $y_i^0(\mathbf{x}) = z_{i,l}(\mathbf{x}) - h_{il}x_l < 0$, and thus $\mathcal{I}_-(\mathbf{x})$ is nonempty. In addition,

$$\begin{aligned}\lambda_i(\mathbf{x}) &\geq \frac{1}{2^{n-1}}(Q(z_{i,l}(\mathbf{x}) - h_{il}x_l) + Q(z_{i,l}(\mathbf{x}) + h_{il}x_l))\\ &\geq \frac{1}{2^{n-1}}(Q(-h_{il}x_l) + Q(h_{il}x_l)) = \frac{1}{2^{n-1}}.\end{aligned}$$

■

Lemma 3. *Given $\mathbf{x} \in \mathcal{X}_1$, $y_i^k(\mathbf{x}) > |y_i^0(\mathbf{x})|$ for all $k \geq 1$ and all i.*

Proof: Fix a user i. For any $k \geq 1$, there exists $l \neq i$ such that $b_l = 1$. Then

$$\begin{aligned}y_i^k(\mathbf{x}) &= h_{ii}x_i + \sum_{j\neq i,l} b_j h_{ij}x_j + h_{il}x_l\\ &\geq h_{ii}x_i - \sum_{j\neq i,l} h_{ij}x_j + h_{il}x_l = z_{i,l}(\mathbf{x}) + h_{il}x_l. \quad (25)\end{aligned}$$

Since $\mathbf{x} \in \mathcal{X}_1$, $z_{i,l}(\mathbf{x}) > 0$. Thus $z_{i,l}(\mathbf{x}) + h_{il}x_l > -z_{i,l}(\mathbf{x}) + h_{il}x_l = -y_i^0(\mathbf{x})$. Moreover, $z_{i,l}(\mathbf{x}) + h_{il}x_l > z_{i,l}(\mathbf{x}) - h_{il}x_l = y_i^0(\mathbf{x})$. Hence, $z_{i,l}(\mathbf{x}) + h_{il}x_l > |y_i^0(\mathbf{x})|$. Together with (25), $y_i^k(\mathbf{x}) > |y_i^0(\mathbf{x})|$ for any $k \geq 1$. ■

Lemma 4. *Given $\mathbf{x} \in \mathcal{X}_1$, for user $i \in \mathcal{I}_-(\mathbf{x})$, there exits α_0 such that when $0 < \alpha < \alpha_0$, $\frac{\mathrm{d}\lambda_i(\alpha\mathbf{x})}{\mathrm{d}\alpha} < 0$ and when $\alpha > \alpha_0$, $\frac{\mathrm{d}\lambda_i(\alpha\mathbf{x})}{\mathrm{d}\alpha} > 0$. Moreover, $\lim_{\alpha\to\infty} \lambda_i(\alpha\mathbf{x}) = \frac{1}{2^{n-1}}$.*

Proof: Consider $\lambda_i(\alpha\mathbf{x})$ where $\alpha \geq 0$ is the variable. From (23),

$$\lambda_i(\alpha\mathbf{x}) = \frac{1}{2^{n-1}} \sum_{k=0}^{2^{n-1}-1} Q\left(\alpha y_i^k(\mathbf{x})\right). \quad (26)$$

Differentiating it with respect to α, we have

$$\begin{aligned}\frac{\mathrm{d}\lambda_i(\alpha\mathbf{x})}{\mathrm{d}\alpha} &= \frac{-1}{2^{n-1}\sqrt{2\pi}} \sum_{k=0}^{2^{n-1}-1} y_i^k(\mathbf{x}) \cdot \exp\left(-\frac{(y_i^k(\mathbf{x})\alpha)^2}{2}\right)\\ &= \frac{-1}{2^{n-1}\sqrt{2\pi}} \exp\left(-\frac{(y_i^0(\mathbf{x})\alpha)^2}{2}\right) \left(\sum_{k\geq 1} y_i^k(\mathbf{x}) \cdot \right.\\ &\quad \left. \exp\left(-\frac{(y_i^k(\mathbf{x})^2 - y_i^0(\mathbf{x})^2)\alpha^2}{2}\right) + y_i^0(\mathbf{x})\right).\end{aligned}$$

Let

$$\Psi(\alpha) = \sum_{k\geq 1} y_i^k(\mathbf{x}) \cdot \exp\left(-\frac{(y_i^k(\mathbf{x})^2 - y_i^0(\mathbf{x})^2)\alpha^2}{2}\right).$$

Since $\mathbf{x} \in \mathcal{X}_1$, by Lemma 3, for $k \geq 1$, $y_i^k(\mathbf{x}) > 0$ and $y_i^k(\mathbf{x})^2 - y_i^0(\mathbf{x})^2 > 0$. Hence $\Psi(\alpha)$ is strictly and monotonically decreasing. Moreover, $\Psi(0) = \sum_{k\geq 1} y_i^k(\mathbf{x}) > -y_i^0(\mathbf{x})$ and $\lim_{\alpha\to\infty} \Psi(\alpha) = 0 < -y_i^0(\mathbf{x})$, where $y_i^0(\mathbf{x}) < 0$ due to $i \in \mathcal{I}_-(\mathbf{x})$. Since $\Psi(\alpha)$ is continuous, there exists α_0 such that $\Psi(\alpha_0) + y_i^0(\mathbf{x}) = 0$. When $0 < \alpha < \alpha_0$, $\frac{\mathrm{d}\lambda_i(\alpha\mathbf{x})}{\mathrm{d}\alpha} = \frac{-1}{2^{n-1}\sqrt{2\pi}} \exp\left(-\frac{(y_i^0(\mathbf{x})\alpha)^2}{2}\right)(\Psi(\alpha) + y_i^0(\mathbf{x})) < 0$ and when $\alpha > \alpha_0$, $\frac{\mathrm{d}\lambda_i(\alpha\mathbf{x})}{\mathrm{d}\alpha} > 0$. In addition,

$$\begin{aligned}\lim_{\alpha\to\infty} \lambda_i(\alpha\mathbf{x}) &= \frac{1}{2^{n-1}} \sum_{k=1}^{2^{n-1}-1} \lim_{\alpha\to\infty} Q(\alpha y_i^k(\mathbf{x}))\\ &\quad + \frac{1}{2^{n-1}} \lim_{\alpha\to\infty} Q(\alpha y_i^0(\mathbf{x})) = 0 + \frac{1}{2^{n-1}} \cdot 1 = \frac{1}{2^{n-1}}.\end{aligned}$$

■

Theorem 3. *Under the non-Gaussian model, for an n-user wireless communication system with synchronized transmission, when the character matrix is not an M-matrix, if $\mathcal{X}_1 = \emptyset$, the minimal BEP $\hat{\lambda} \geq \frac{1}{2^{n-1}}$; if $\mathcal{X}_0 = \emptyset$ but $\mathcal{X}_1 \neq \emptyset$, $\hat{\lambda} = \inf_{\mathbf{x}\in\mathcal{X}_1} \max_{i=1,\ldots,n} \lambda_i(\mathbf{x})$ and $\frac{1}{2^n} < \hat{\lambda} < \frac{1}{2^{n-1}}$.*

Proof: When $\mathcal{X}_1 = \emptyset$, by Lemma 2, $\hat{\lambda} = \inf_{\mathbf{x}\notin\mathcal{X}_1} \max_{i=1,\ldots,n} \lambda_i(\mathbf{x}) \geq \frac{1}{2^{n-1}}$.

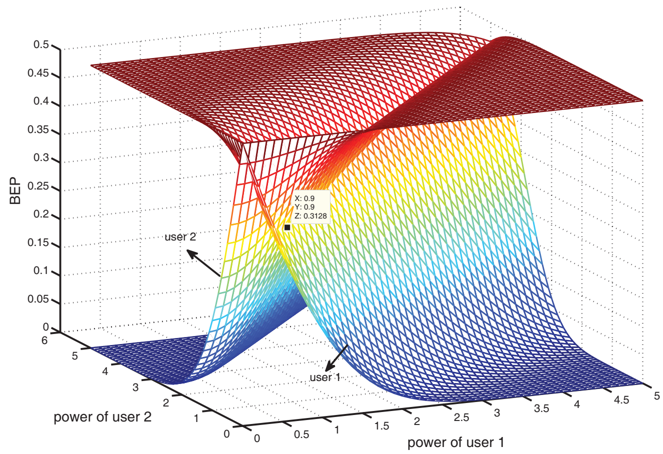

Fig. 2. BEP versus the power of user 1 and user 2. The solution to the minimal BEP problem is achieved by a finite power vector.

When $\mathcal{X}_1 \neq \emptyset$, consider $\tilde{\mathbf{x}} \in \mathcal{X}_1$. Since the character matrix is not an M-matrix and $\mathcal{X}_0 = \emptyset$, by Lemma 1, there exists i such that $y_i^0(\tilde{\mathbf{x}}) < 0$, and thus $i \in \mathcal{I}_-(\tilde{\mathbf{x}}) \neq \emptyset$. By Lemma 4, for user $i \in \mathcal{I}_-(\tilde{\mathbf{x}})$ there exists α_i such that $\frac{\mathrm{d}\lambda_i(\alpha\tilde{\mathbf{x}})}{\mathrm{d}\alpha} > 0$ when $\alpha > \alpha_i$ and $\lim_{\alpha\to\infty} \lambda_i(\alpha\tilde{\mathbf{x}}) = \frac{1}{2^{n-1}}$. So we have $\lambda_i(\alpha\tilde{\mathbf{x}}) < \frac{1}{2^{n-1}}$ for $\alpha \geq \alpha_i$. For $i' \in \overline{\mathcal{I}}_-(\tilde{\mathbf{x}})$, since $y_{i'}^k(\tilde{\mathbf{x}}) \geq 0$ for all k, $\lambda_{i'}(\alpha\tilde{\mathbf{x}})$ is monotonically decreasing with α and $\lim_{\alpha\to\infty} \lambda_{i'}(\alpha\tilde{\mathbf{x}}) \leq \frac{1}{2^n}$. Therefore there exists $\alpha_{i'}$ such that $\lambda_{i'}(\alpha\tilde{\mathbf{x}}) < \frac{1}{2^{n-1}}$ when $\alpha \geq \alpha_{i'}$. Let $\alpha^* = \max_i \alpha_i$. Then for $\alpha \geq \alpha^*$, $\max_i \lambda_i(\alpha\tilde{\mathbf{x}}) < \frac{1}{2^{n-1}}$. So

$$\begin{aligned}\hat{\lambda} &= \min\left\{ \inf_{\mathbf{x}\notin\mathcal{X}_1} \max_{i=1,\ldots,n} \lambda_i(\mathbf{x}), \inf_{\mathbf{x}\in\mathcal{X}_1} \max_{i=1,\ldots,n} \lambda_i(\mathbf{x}) \right\} \\ &= \inf_{\mathbf{x}\in\mathcal{X}_1} \max_{i=1,\ldots,n} \lambda_i(\mathbf{x}) \leq \max_{i=1,\ldots,n} \lambda_i(\alpha^*\tilde{\mathbf{x}}) < \frac{1}{2^{n-1}}.\end{aligned}$$

Moreover, since there exists i such that $y_i^0(\mathbf{x}) < 0$, the second inequality in (24) holds strictly. Therefore $\hat{\lambda} > \frac{1}{2^n}$. ∎

Fig. 2 plots the BEP versus power for a two-user system with synchronous transmission. The channel gain matrix is symmetric and $h_{11} < h_{12}$. So the character matrix is a non-M-matrix. It can be found that the minimal BEP 0.3128 is achieved by a finite power vector (0.9, 0.9). To conclude, the relations of the minimal BEP and the character matrix are summarized in Table I. From the table, the limitation of the Gaussian model in predicting the BEP becomes clear.

V. The Minimal Power Problem

In previous discussion, we focus on the minimal BEP problem, which provides a theoretic bound on the system performance. In this section, we consider another common power control problem: minimize the total transmission power while maintaining an acceptable QoS for each user, where the QoS is in terms of BEP. Our purpose is to further explore the property of the BEP function $\lambda_i(\mathbf{x})$ given by (7) under the non-Gaussian model and to compare the power control results under the non-Gaussian model with those under the Gaussian model. Let ϵ be the target BEP, the minimal power problem can be stated as

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} x_i^2 \\ \text{s.t.} \quad & \lambda_i(\mathbf{x}) \leq \epsilon \qquad i = 1,\ldots,n. \end{aligned} \tag{27}$$

Under the Gaussian model, by (11), we see that $\lambda_i^G(\mathbf{x})$ is monotonically decreasing with SINR. Thus the requirement of BEP $\lambda_i^G(\mathbf{x}) \leq \epsilon$ can be transformed to the requirement of SINR $\gamma_i(\mathbf{x}) \geq \Gamma$, where $\epsilon = Q(\sqrt{\Gamma})$. It is exactly equivalent to the SINR tracking problem: $\min \sum_i x_i^2$ s.t. $\gamma_i(\mathbf{x}) \geq \Gamma$, which can be solved via linear programming. Let $\mathbf{u}$ be the normalized noise vector with entry $u_i = \sigma_i^2/T_i^2 h_{ii}^2$. $\mathbf{Z} = (\frac{h_{ij}^2}{h_{ii}^2})$ is the normalized power gain matrix as defined in Section III. $\mathbf{I}$ is an $n \times n$ identity matrix. If the Perron-Frobenius eigenvalue of $\mathbf{Z}$ satisfies $\rho_{\mathbf{Z}} < 1 + \frac{1}{\Gamma}$, the optimal power setting is [4]

$$[x_1^2, \ldots, x_n^2]^\top = [(\frac{1}{\Gamma} + 1)\mathbf{I} - \mathbf{Z}]^{-1}\mathbf{u}. \tag{28}$$

Under the non-Gaussian model, the minimal power problem (27) cannot be transformed to a linear programming. Moreover, since $\lambda_i(\mathbf{x})$ is a linear combination of Q-functions and Q-function is non-convex on $\mathbb{R}$, $\lambda_i(\mathbf{x})$ is in general non-convex on $\{\mathbf{x} : \mathbf{x} > 0\}$, and thus (27) is not a convex optimization. However, note that the Q-function is convex on $\mathbb{R}^+$. We can add a constraint to ensure the new problem is convex. Define

$$\mathcal{X}_\epsilon = \{\mathbf{x} : \lambda_i(\mathbf{x}) \leq \epsilon, i = 1,\ldots,n\}. \tag{29}$$

Proposition 1. *There exists ϵ_0 such that for $\epsilon \leq \epsilon_0$, $\mathcal{X}_\epsilon \subset \mathcal{X}_{\mathbf{C}}$.*

Proof: Let $\epsilon \leq \epsilon_0 \triangleq \min_{i=1,\ldots,n}(P_i/2)$, where P_i is defined in (20). If there exists an $\mathbf{x} \in \mathcal{X}_\epsilon$ but $\mathbf{x} \notin \mathcal{X}_{\mathbf{C}}$, at least one entry in $\mathbf{Cx}$, say the ith entry, is not positive. By (21), $\lambda_i(\mathbf{x}) > P_i/2 \geq \epsilon_0 \geq \epsilon$, i.e., a contradiction to $\mathbf{x} \in \mathcal{X}_\epsilon$. Hence $\mathcal{X}_\epsilon \subset \mathcal{X}_{\mathbf{C}}$. ∎

In practice, an acceptable BEP requirement rarely can be higher than 10^{-3}, e.g., the tolerable uncoded BEP is less than 10^{-5} for data and 10^{-6} for video [17]. We consider a system of less than six users. It is reasonable to assume $\epsilon < 10^{-3} < \epsilon_0$. It needs to be mentioned that, without this assumption, (27) can be solved following a similar method to be discussed later, but at the expense of defining more cumbersome parameters. In this paper, we only present how to solve (27) with the assumption. So (27) is equivalent to

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} x_i^2 \\ \text{s.t.} \quad & \lambda_i(\mathbf{x}) \leq \epsilon \qquad i = 1,\ldots,n \\ & \mathbf{x} \in \mathcal{X}_{\mathbf{C}}. \end{aligned} \tag{30}$$

If the character matrix $\mathbf{C}$ is not an M-matrix, $\mathcal{X}_{\mathbf{C}} = \emptyset$ and there is no feasible solution to (30). It is easy to explain: when $\mathbf{C}$ is not an M-matrix, given any amplitude setting, there exists a user whose signal is weaker than the aggregated interference. Thus the BEP of that user fails to satisfy the QoS requirement. In that case, a scheduling is required to select candidate subsets of concurrently active users and this investigation is out of the scope of this paper. In subsequent

TABLE I
COMPARISON BETWEEN THE NON-GAUSSIAN MODEL AND GAUSSIAN MODEL

Character matrix		minimal BEP (non-Gaussian)	minimal BEP (Gaussian)
M-matrix ($\mathcal{X}_{\mathbf{C}} \neq \emptyset$)		$\hat{\lambda} = 0$	$\hat{\lambda}^G = Q\left(\sqrt{\frac{1}{\rho_{\mathbf{Z}}-1}}\right)$
Non M-matrix (synchronized transmission)	$\mathcal{X}_1 = \emptyset$	$\hat{\lambda} \geq 1/2^{n-1}$	
	$\mathcal{X}_0 \neq \emptyset$	$\hat{\lambda} = 1/2^n$	
	$\mathcal{X}_0 = \emptyset, \mathcal{X}_1 \neq \emptyset$	$1/2^n < \hat{\lambda} < 1/2^{n-1}$	

discussion, we consider that $\mathbf{C}$ is an M-matrix. So $\mathcal{X}_{\mathbf{C}} \neq \emptyset$. Moreover, by Theorem 1, $\mathcal{X}_\epsilon \neq \emptyset$.

Remark: when $\mathbf{C}$ is an M-matrix, $\mathbf{C}' = (c_{ij} + \Delta_{ij})$ is still an M-matrix if $\max_{i,j}\{|\Delta_{ij}|\}$ is small enough. Hence for fading channel, if the perturbations of the channel gains are small, the M-matrix structure of the character matrix is preserved. Therefore the feasibility of the minimal power problem (30) is robust to slow and flat fading up to some extent.

The following three Lemmas describe the properties of $\lambda_i(\mathbf{x})$ over $\mathcal{X}_{\mathbf{C}}$.

Lemma 5. *For any user i, $\lambda_i(\mathbf{x})$ is convex on $\mathcal{X}_{\mathbf{C}}$.*

Proof: $\mathcal{X}_{\mathbf{C}}$ is an intersection of n half-spaces and hence convex. Fix a user i and assume $b_i^0 = 1$. Let $q_{ij} = h_{ij}\int_0^{T_i} a_j(t-\tau_{ij})\mathrm{d}t$ and $\mathbf{q}_i = [q_{i1}, \dots, q_{in}]$. Given $\mathbf{q}_i$, by (5), $\sum_{j=1}^n W_{ij} = \mathbf{q}_i\mathbf{x}$ is a linear function of $\mathbf{x}$. So the range of $\sum_{j=1}^n W_{ij}$ over $\mathcal{X}_{\mathbf{C}}$ is convex. For $\mathbf{x} \in \mathcal{X}_{\mathbf{C}}$, we have

$$\sum_{j=1}^n W_{ij} = b_i^0 T_i h_{ii} x_i + \sum_{j\neq i} W_{ij} \geq T_i h_{ii} x_i - T_i \sum_{j\neq i} h_{ij} x_j > 0.$$

Since $Q(x)$ is convex on $x > 0$, by the composition rule of convexity-preserving, $Q(\sum_{j=1}^n W_{ij}/\sigma_i) = Q(\mathbf{q}_i\mathbf{x}/\sigma_i)$ is convex on $\mathcal{X}_{\mathbf{C}}$. Therefore $\lambda_i(\mathbf{x}) = \mathrm{E}_{\mathbf{q}_i}[Q(\mathbf{q}_i\mathbf{x}/\sigma_i)]$ in (7) is convex on $\mathcal{X}_{\mathbf{C}}$. ■

By Lemma 5, we see that problem (30) is convex. For the convex optimization problem with strictly convex objective function, there exists at most one global minimizer [18]. Hence, if the optimal value of (30) can be attained, the minimizer is unique.

Lemma 6. *For any user i, over $\mathbf{x} \in \mathcal{X}_{\mathbf{C}}$, $\lambda_i(\mathbf{x})$ is a strictly and monotonically decreasing function of x_i and a strictly and monotonically increasing function of x_k for $k \neq i$.*

The proof of Lemma 6 can be found in [15]. The idea is to prove $\frac{\partial \lambda_i(\mathbf{x})}{\partial x_i} < 0$ and $\frac{\partial \lambda_i(\mathbf{x})}{\partial x_k} > 0$.

Lemma 7. *Define a function $\lambda(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^n$ with component functions $\lambda_1(\mathbf{x}), \dots, \lambda_n(\mathbf{x})$. Then $\lambda(\mathbf{x})$ is injective on $\mathcal{X}_{\mathbf{C}}$.*

Proof: We first consider that the channel gain matrix $\mathbf{H} = (h_{ij})$ is row diagonally dominant. In this case, the character matrix $\mathbf{C}$ is an M-matrix and $\mathcal{X}_{\mathbf{C}} \neq \emptyset$. Suppose $\mathbf{x}', \hat{\mathbf{x}} \in \mathcal{X}_{\mathbf{C}}$ and $\mathbf{x}' \neq \hat{\mathbf{x}}$. Let $\Delta\mathbf{x} = \hat{\mathbf{x}} - \mathbf{x}'$. Suppose $l \in \arg\max_i\{|\Delta x_i|\}$. Since $\mathbf{x}' \neq \hat{\mathbf{x}}$, $\Delta x_l \neq 0$. Suppose $\Delta x_l < 0$, otherwise exchange the value of $\hat{\mathbf{x}}$ and $\mathbf{x}'$. By Lemma 5, $\lambda_l(\mathbf{x})$ is convex on $\mathcal{X}_{\mathbf{C}}$, and therefore

$$\lambda_l(\hat{\mathbf{x}}) \geq \lambda_l(\mathbf{x}') + \nabla\lambda_l(\mathbf{x}')^\top(\hat{\mathbf{x}} - \mathbf{x}'). \tag{31}$$

If $\nabla\lambda_l(\mathbf{x}')^\top(\hat{\mathbf{x}} - \mathbf{x}') > 0$, we have $\lambda_l(\hat{\mathbf{x}}) > \lambda_l(\mathbf{x}')$. Then for any $\mathbf{x}' \neq \hat{\mathbf{x}}$, $\lambda(\mathbf{x}') \neq \lambda(\hat{\mathbf{x}})$ and thus $\lambda(\mathbf{x})$ is injective on $\mathcal{X}_{\mathbf{C}}$. Now we show the proof. The q_{lj} is defined in the proof of Lemma 5.

$$\begin{aligned}
&\nabla\lambda_l(\mathbf{x}')^\top(\hat{\mathbf{x}} - \mathbf{x}') = \sum_{j=1}^n \frac{\partial \lambda_l(\mathbf{x}')}{\partial x_j}\Delta x_j \\
&= \sum_{j=1}^n \mathrm{E}_{\substack{q_{lj'}\\ j'\neq l}}\left[\frac{-q_{lj}}{\sigma_l\sqrt{2\pi}}\exp\left(-\frac{(\sum_{j=1}^n q_{lj}x'_j)^2}{2\sigma_l^2}\right)\right]\cdot\Delta x_j \\
&= \mathrm{E}_{\substack{q_{lj'}\\ j'\neq l}}\left[\frac{1}{\sigma_l\sqrt{2\pi}}\left(\sum_{j=1}^n -q_{lj}\Delta x_j\right)\exp\left(-\frac{(\sum_{j=1}^n q_{lj}x'_j)^2}{2\sigma_l^2}\right)\right] \\
&> 0.
\end{aligned}$$

The last inequality holds since for any q_{lj}, we have,

$$\begin{aligned}
\sum_{j=1}^n q_{lj}\Delta x_j &\overset{(a)}{\leq} h_{ll}T_l\Delta x_l + \sum_{j\neq l} h_{lj}T_l|\Delta x_l| \\
&= T_l\Delta x_l\left(h_{ll} + \sum_{j\neq l} -h_{lj}\right) \overset{(b)}{<} 0,
\end{aligned}$$

where (a) holds since $|\Delta x_l| \geq \Delta x_j$ for all j and (b) holds since $\mathbf{H}$ is row diagonally dominant.

Now we consider that $\mathbf{H}$ is not row diagonally dominant while $\mathbf{C}$ is an M-matrix. By the fourth equivalent condition of M-matrix in Definition 3, there exists a positive diagonal matrix $\mathbf{D} = \mathrm{diag}(d_1, \dots, d_n)$, such that $\mathbf{CD}$ is strictly row diagonally dominant. Hence, $d_i h_{ii} > \sum_{j\neq i} h_{ij} d_j$ for all i. Let $\mathbf{H}' = \mathbf{HD}$. So $\mathbf{H}'$ is row diagonally dominant. Let $\mathbf{C}_{\mathbf{H}'}$ denote the character matrix of $\mathbf{H}'$. We can see that $\mathbf{C}_{\mathbf{H}'} = \mathbf{CD}$ and $\mathcal{X}_{\mathbf{C}_{\mathbf{H}'}} = \{\mathbf{x} : \mathbf{C}_{\mathbf{H}'}\mathbf{x} > \mathbf{0}\} = \{\mathbf{D}^{-1}\mathbf{x} : \mathbf{x} \in \mathcal{X}_{\mathbf{C}}\}$. $\lambda_{\mathbf{H}'}(\mathbf{x})$ and $\lambda(\mathbf{x})$ are the BEP functions for $\mathbf{H}'$ and $\mathbf{H}$ respectively. Since $\lambda_{\mathbf{H}'}(\mathbf{x})$ is injective on $\mathcal{X}_{\mathbf{C}_{\mathbf{H}'}}$, $\lambda(\mathbf{x}) = \lambda_{\mathbf{H}'}(\mathbf{D}^{-1}\mathbf{x})$ is injective on $\mathcal{X}_{\mathbf{C}}$. ■

Next, we propose an algorithm to prove that the optimal value of (30) is attainable and the minimizer $\mathbf{x}^*$ satisfies the inequality constraints with equality, i.e., $\lambda_i(\mathbf{x}^*) = \epsilon$ for $i = 1, \dots, n$. **Iterative Descent Algorithm:**

Input $\mathbf{x}^{(0)} \in \mathcal{X}_\epsilon$ and set $k = 1$.

1) $i = |k|_{\bmod n}$.
   If $\lambda_i(\mathbf{x}^{(k)}) < \epsilon$,
   let $\mathbf{x}^{(k+1)} = [x_1^{(k)}, \dots, x_{i-1}^{(k)}, x_i^{(k+1)}, x_{i+1}^{(k)}, \dots, x_n^{(k)}]$ s.t. $\lambda_i(\mathbf{x}^{(k+1)}) = \epsilon$;
   otherwise $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)}$.
2) $k \leftarrow k + 1$, go to 1).

Lemma 8. *For each user i, the Iterative Descent Algorithm generates a sequence $\{x_i^{(k)}\}_k$. When $\epsilon \leq \epsilon_0$, the sequence $\{x_i^{(k)}\}_k$ is monotonically decreasing and is bounded below* *by zero, thus it is convergent. Suppose $\{x_i^{(k)}\}_k$ converges to $\widetilde{x}_i$. Let $\widetilde{\mathbf{x}} = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$. Then $\lambda_i(\widetilde{\mathbf{x}}) = \epsilon$ for all i.*

Proof: When $\epsilon \leq \epsilon_0$, the input $\mathbf{x}^{(0)} \in \mathcal{X}_\epsilon \subset \mathcal{X}_\mathbf{C}$ (ref. Prop. 1). If $\lambda_i(\mathbf{x}^{(0)}) = \epsilon$ for all i, then we are done. Otherwise $\lambda_i(\mathbf{x}^{(0)}) < \epsilon$ for some i. We prove that the sequences $\{x_i^{(k)}\}_k$ for $i = 1, \ldots, n$ are monotonically decreasing by mathematical induction. At the first step, $x_1^{(0)}$ is updated. Given $x_j^{(0)}$, $j \neq 1$, $\lambda_1(x_1, x_2^{(0)}, \ldots, x_n^{(0)})$ is a continuous and monotonically decreasing function of x_1. Since $\lambda_1(\mathbf{x}^{(0)}) \leq \epsilon$ and $\lambda_1(0, x_2^{(0)}, \ldots, x_n^{(k)}) = 1/2 > \epsilon$, there exists $x_1^{(1)}$ s.t. $\lambda_1(\mathbf{x}^{(1)}) = \epsilon$ and $0 < x_1^{(1)} \leq x_1^{(0)}$. So $\mathbf{0} < \mathbf{x}^{(1)} \leq \mathbf{x}^{(0)}$. As $\mathbf{x}^{(0)} \in \mathcal{X}_\mathbf{C}$, it can be checked that $\mathbf{x}^{(1)} \in \mathcal{X}_\mathbf{C}$. Further by Lemma 6, $\lambda_j(\mathbf{x})$ increases with x_1, therefore $\lambda_j(\mathbf{x}^{(1)}) \leq \lambda_j(\mathbf{x}^{(0)}) \leq \epsilon$ for $j \neq 1$. So $\mathbf{x}^{(1)} \in \mathcal{X}_\epsilon$. Suppose the statement is truth for k steps. That is $\mathbf{0} < \mathbf{x}^{(k)} \leq \mathbf{x}^{(k-1)} \leq \cdots \leq \mathbf{x}^{(0)}$ and $\mathbf{x}^{(m)} \in \mathcal{X}_\epsilon \subset \mathcal{X}_\mathbf{C}$ for $m \leq k$. At the $k+1$th step, $x_i^{(k)}$ is updated, where $i = |k+1|_{\bmod n}$. Following the same arguments, we have $\mathbf{0} < \mathbf{x}^{(k+1)} \leq \mathbf{x}^{(k)}$ and $\mathbf{x}^{(k+1)} \in \mathcal{X}_\epsilon$. Now we already prove that the sequences $\{x_i^{(k)}\}_k$ are monotonically decreasing and are bounded below by zero.

Let $\widetilde{\mathbf{x}} = \lim_{k\to\infty} \mathbf{x}^{(k)}$. For any arbitrarily small $\delta > 0$, since $\lambda_i(\mathbf{x})$'s are continuous functions, there exists a sufficiently large K, when $k > K$ we have

$$\left|\lambda_i(\widetilde{\mathbf{x}}) - \lambda_i(\mathbf{x}^{(k)})\right| < \delta \qquad \forall i = 1, \ldots, n. \tag{32}$$

By the algorithm, $\lambda_i(\mathbf{x}^{(k')}) = \epsilon$ for some $k' > K$. Therefore, from (32), we have

$$|\lambda_i(\widetilde{\mathbf{x}}) - \epsilon| < \delta \qquad \forall i = 1, \ldots, n. \tag{33}$$

That is, $\lambda_i(\widetilde{\mathbf{x}}) = \epsilon$ for all i. ■

Theorem 4. *The optimal value of* (30) *is attainable and the minimizer $\mathbf{x}^*$ satisfies the inequality constraints with equality, i.e., $\lambda_i(\mathbf{x}^*) = \epsilon$ for $i = 1, \ldots, n$. Given any initial point $\mathbf{x}^{(0)} \in \mathcal{X}_\epsilon$, the Iterative Descent Algorithm converges to the global minimizer $\mathbf{x}^*$.*

Proof: For any $\mathbf{x} \in \mathcal{X}_\epsilon$ with $\lambda_i(\mathbf{x}) < \epsilon$ for some i, applying the Iterative Descent Algorithm with $\mathbf{x}^{(0)} = \mathbf{x}$, by Lemma 8, we obtain monotonically decreasing sequences $\{x_i^{(k)}\}_k$ which convergence to $\widetilde{x}_i$ for all i and $\lambda_i(\widetilde{\mathbf{x}}) = \epsilon$ for all i. It is seen that $\sum_i \widetilde{x}_i^2 \leq \sum_i x_i^2$. As a result, the inequality constraint in (30) is satisfied with the equality $\lambda_i(\mathbf{x}) = \epsilon$ for all i. By Lemma 8, the solution to this equality exists and by Lemma 7, the solution is unique, denoted by $\mathbf{x}^*$. Since $\mathbf{x}^*$ is bounded, the optimal value of (30) is attainable. Besides, since $\mathbf{x}^*$ is unique, given any initial point $\mathbf{x}^{(0)} \in \mathcal{X}_\epsilon$, the Iterative Descent Algorithm converges to the global minimizer $\mathbf{x}^*$. ■

Remark: First, for the simplicity of analysis, in (27), the target BEP of all users are set to be the same ϵ. However, it can be generalized to each user having an individual ϵ_i, and all the conclusions in this section hold with slight modification. Second, if the objective function in (27) is changed to any function $f(\mathbf{x})$ satisfying $f(\mathbf{x}) \geq f(\mathbf{x}')$ when $\mathbf{x} \geq \mathbf{x}'$, the conclusions in this section hold. This property of the objective function is utilized in the proof of Theorem 4.

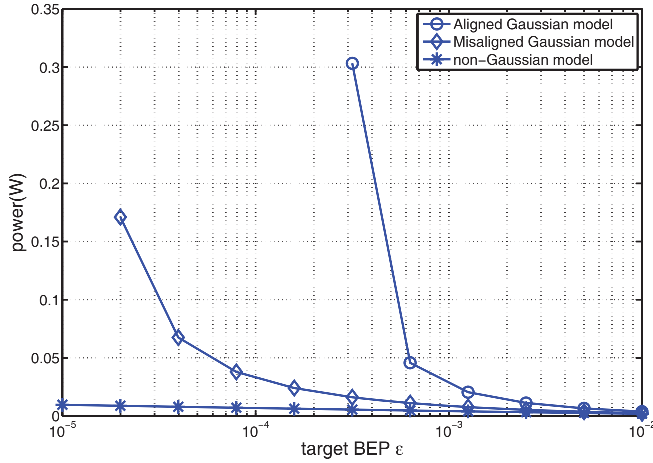

Fig. 3. Power as a function of target BEP ϵ.

VI. Simulation Results

Simulation studies were performed for two-user and three-user systems. For all i and j, the channel gain h_{ij} is drawn from a uniform distribution on the interval $[0, 1]$. We select the channel gain matrix whose character matrix is an M-matrix. The uniform distribution is used here only for simplicity. It is not a necessary condition for the following discussion to be valid. The power spectral density N_0 of the AWGN is 10^{-10} W/Hz. The basic transmission bit rate $R = 1$Mb/s.

First, we compare the results of the minimal power problem under the Gaussian models and non-Gaussian model. Consider a system involving three active users and each transmitter transmits at the same bit rate R. Fig. 3 plots the power versus target BEP ϵ for one of the users. The power under the non-Gaussian model are found by the Iterative Descent Algorithm. The solutions under the Gaussian models are solved by (28). It is seen that the required powers under the Gaussian models deviate from those under the non-Gaussian model. Roughly, the smaller the target ϵ, the lager the differences. Besides, under the Aligned Gaussian model, $\epsilon \leq 10^{-3.6}$ is not achievable and the required power tend to go to infinity as ϵ is close to $10^{-3.6}$. This matches the theoretical value of the minimal BEP $\hat{\lambda}^G = 10^{-3.57}$, calculated by (13). The Misaligned Gaussian model has similar performance. In conclusion, under the Gaussian models, the BEP of all users cannot be arbitrarily small.

Next, we investigate a two-user system where the transmitters use different bit rates. For the convenience of illustration, we set $h_{11} = h_{22}$ and $h_{12} = h_{21}$. The power setting is $x_1^2 = x_2^2 = 0.3$mW. The bit rate of user 1 is fixed at $R_1 = R$, while the bit rate of user 2, R_2, changes from R to $R/10$. Fig. 4 shows the effect of varying bit rate on BEP under different models. Under the three models, the BEP of user 2 all decrease as R_2 decreases. One of the reasons is that the normalized noise variance $\sigma_2^2/T_2^2 = N_0 R_2/2$ decreases with R_2. For the Misaligned Gaussian model and non-Gaussian model, there is another reason, that is, the normalized interference variance $\sigma_{W_{21}}^2/T_2^2$ (see (9)) also decreases with R_2. On the other hand, the Aligned Gaussian model assumes constant interference variance, so the decrement under this model is not so significant. As for user 1, under the Aligned Gaussian

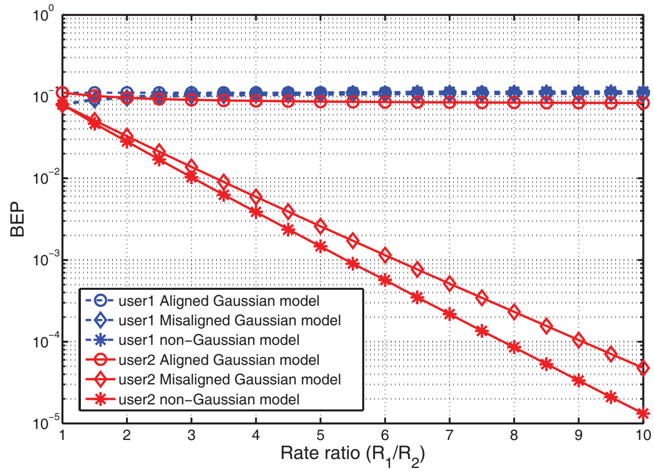

Fig. 4. BEP as a function of bit rate ratio (R_1/R_2). $R_1 = R$ is fixed. R_2 decreases from R to $R/10$. The dashed lines are for user1, the solid lines are for user2.

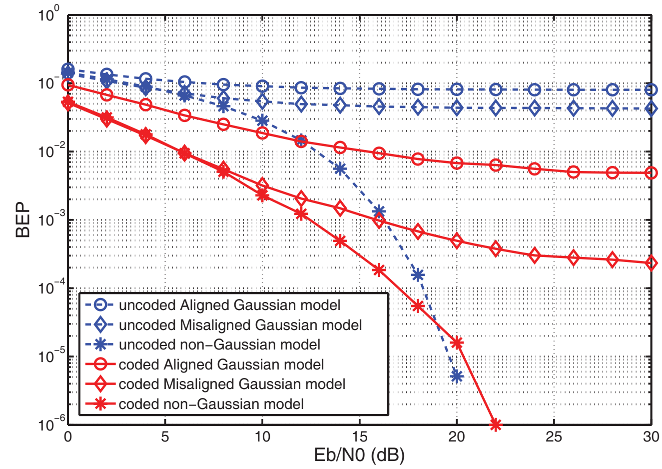

Fig. 5. BEP as a function of E_b/N_0. The dashed lines are for the uncoded system, the solid lines are for the coded system.

model, the BEP is invariant, while under the Misaligned Gaussian model and non-Gaussian model, the BEP increase as R_2 decreases. This is because $\sigma^2_{W_{12}}/T_1^2$ increases as R_2 decreases. The Misaligned Gaussian model has similar results with the non-Gaussian model, but as it assumes that the interference is Gaussian distributed, the effect of varying bit rate on BEP is less significant.

Until now, we have been discussing an uncoded system. Next, we report some simulations performed for a coded two-user system. In these simulations a convolutional code with coding rate $1/2$ and Viterbi-decoding with soft-decision was used. The modulation scheme is BPSK. We set $h_{11} = h_{22}$, $h_{12} = h_{21}$, fix N_0 and the symbol rates by $R_1 = R_2 = R$. We set the symbol amplitudes by $x_1 = x_2 = x$ and select x such that $\frac{E_b}{N_0} = \frac{2E_s}{N_0} = \frac{2x^2/R}{N_0}$ with $\frac{E_b}{N_0}$ ranging from 0dB to 30dB. Fig. 5 illustrates BEP versus $\frac{E_b}{N_0}$ for a coded system and an uncoded system. In both cases, the Gaussian models yield pessimistic results. Under the Gaussian models, the BEP cannot be arbitrarily small, while under the non-Gaussian model, the BEP can decrease to zero. Moreover, the Misaligned Gaussian model and the non-Gaussian model start out to have similar performance for small $\frac{E_b}{N_0}$ ratio and diverge significantly when the ratio increases. However, for the uncoded system the divergence starts to occur for smaller values of $\frac{E_b}{N_0}$ (around 6dB) when compared to the coded system (around 8db).

VII. Concluding Remarks

In this paper, we investigate a non-Gaussian interference model. As demonstrated by both analysis and simulations, the non-Gaussian model has significantly different performance characteristics from the Gaussian model. We note that the analysis in this study, although specific to certain assumptions, can be extended to more general situations, for example when the channels are time-varying. However, the results reported in the paper aim to shed further lights on more realistic interference models and may lead to more effective power control algorithms.

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Yi Chen received the B.S. degree in Communication Engineering in 2007 from Beijing University of Posts and Telecommunications, Beijing, China. She is currently pursuing the Ph.D. degree in Information Engineering at the Chinese University of Hong Kong. Her research interests include resource allocation, power control and interference cancellation for wireless communication systems.

Wing Shing Wong (M'81–SM'90–F'02) received a combined master and bachelor degree from Yale University and M.S. and Ph.D. degrees from Harvard University. He joined AT&T Bell Laboratories in 1982 and moved to the Chinese University of Hong Kong in 1992, where he is now a Professor of Information Engineering and Dean of the Graduate School. He was the Chairman of the Department of Information Engineering from 1995 to 2003. He served as Science Advisor at the Innovation and Technology Commission of the HKSAR government from 2003 to 2005 and was a Board Director and Chairman of the Technical Committee of the Hong Kong Applied Science and Technology Research Institute. He has participated in a variety of research and development projects on topics ranging from mobile communication, nonlinear filtering, to information based control.