Invariant Distributions of Linear Systems under Finite Communication Bandwidth Feedback

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***Abstract*— In the paper, we study the asymptotic probabilistic behavior of a system stabilized by finite communication bandwidth feedback control in the form of an essentially symmetric 1-bit control law. It is shown that the state orbits eventually converge to an invariant interval under the proposed coded control law. If the resulting closed-loop system is a Markov transformation, the invariant density is piecewise constant and can be associated with the left eigenvector of a non-negative matrix induced by the transformation. The optimal control law that minimizes an asymptotic expected cost function is also derived when the transformation is a covering.**

I. INTRODUCTION

Linear systems with finite communication bandwidth feedback control have received much attention in the last few years [1]-[10]. In this new research area of control theory, the information exchange between the plant and the controller is assumed to be coded and transmitted over digital channels with finite capacity. Various control policies using memory-less [1]-[6] or memory-based [7]-[10] coding have been proposed and analyzed, with an emphasis towards a goal of system stabilization. A closely related circle of ideas analyzing the effect of quantization on the feedback control was reported early in [1], which in particular includes results pertaining to the asymptotic probabilistic behavior of such systems.

It has been known from the work reported in [1] that linear systems with a uniform quantizer cannot be asymptotically stabilized to zero. In [2], a weaker stability concept called containability was introduced, which requires that the initial state converges to a bound open set around the origin. In [3], it was shown that the coarsest quantizer for stabilization of a linear system is logarithmic. In [4], stabilization was established for adaptive quantization schemes. In [5], a tight bound on the channel capacity was derived for data-rate limited systems, and binary control was considered via an interesting special case. In [6], the performance of static memory-less quantizers was analyzed taking into account of the number of quantization levels and the convergence time. The results in [6] were extended to quantizers with a memory structure in [7]. In [9], a time-varying quantized feedback control with memory was proposed using an infinite time horizon quadratic regulation criteria. Explicit formulae for the optimal cost and policy were derived when the initial state is uniformly distributed.

In this paper, we consider a control system which relies on a remotely located observer. For motivation, one can consider an earth-based vehicular system whose coordinates are only measurable by a remote-sensing satellite. (Note that this is different from the model of a GPS system in which the vehicular system can determine its coordinates from the triangulations of broadcast signals from multiple satellites.) If the satellite has to support a large number of such vehicular systems, the amount of communication bandwidth devoted to a single system could be limited. As motivated by the concept of minimum attention control in [11], one can consider a related concept of minimum attention observation so that the data rate for the observation data is minimized. In the extreme case, one can consider the stabilization effect of feedback control laws based on a 1-bit observation scheme. The control law can of course be either memory-less or memory-based. In this paper, only the former case is considered. It is further assumed that the initial system state, which is unknown except to the remote observer, can be characterized by a distribution known to the system. As usual, the focus is on systems which are inherently unstable. A natural question then is to consider optimal or locally optimal control laws based on the 1-bit observation data. If the underlying system is a 1-dimensional linear system then under suitable conditions it can be shown that the trajectory of system stays within a bounded interval. Since the closed-loop system can be represented by an expanding piecewise linear map, it is well-known that the state trajectory behaves chaotically and that its asymptotic behavior can be described by an invariant measure [16]. Lasota and Yorke [12] proved the existence of absolutely continuous invariant measures for expanding piecewise maps. For general dynamical systems, it is possible to compute the invariant density only in very simple cases. Methods of approximating the invariant density were proposed in [13]-[14]. It is shown that the invariant density must be piecewise constant and can be obtained from an induced matrix when the map is an expanding piecewise linear Markov transformation [15]. If the map is not Markov, its invariant distribution can be approximated by those of a sequence of converging, piecewise linear Markov approximations [14].

The work reported here bears close resemblance to the work reported in [1]. There are, however, major differences. In [1], the objective is to understand the effect of quantized state measurements on a feedback control law. The number of quantized states is unbound. In this paper, we aim to

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analyze the optimal feedback law that can be defined based on a 1-bit observation and to understand its asymptotic behavior. In particular, we focus on the computation of the invariant density when the related closed-loop system is a Markov transformation. It should be pointed out that in [5], the invariant distribution for a multivariable system with specialized system parameters was computed explicitly. Here, our approach is based on Markov transformation and applies to a broad class of systems. The construction of a Markov transformation is investigated. Moreover, we derive the optimal control law based on a 1-bit observation scheme.

The rest of the paper is organized as follows: Section II contains the stabilizability analysis of the closed-loop system obtained by using a 1-bit control law. In Section III, the concept of invariant measures is described. In Section IV, the asymptotic dynamics of the system is analyzed in details, and the optimal control law minimizing the asymptotic expected cost function is derived when the closed-loop system is a covering. Concluding remarks are presented in Section V.

II. OPTIMAL FINITE ATTENTION BIT CONTROL LAW

A. Problem Statement

Consider a plant that can be described by a scalar, discrete-time system

$$\left\{ \begin{array}{lll} x_{n+1} & = & ax_n + u(y_n), x_0 \in \Re, \\ y_n & = & c(x_n), \end{array} \right. \tag{1}$$

where x_n and u denote the system state and the control, respectively. It is assumed that $a > 1$, so that the underlying system is unstable. The controller is based on information supplied by a remote observer. The observation data, once obtained, are encoded into y_n by the function $c(\cdot) : \Re \to \mathcal{S}$, where $\mathcal{S}$ denotes the codebook with a size 2^k . The data is then transmitted over a digital channel by a k-bit codeword at each time instance.

Assume that the initial state is prescribed by a probability density function (pdf) f_0. If the control law u is fixed, the state x_1 is distributed according to a new pdf $f_1 = Pf_0$, where P is the Frobenius-Perron operator [12]. In general, if we denote the pdf corresponding to x_n by f_n , then

$$f_n = P^n f_0. \tag{2}$$

If the remote observer has to support a large number of systems, the communication data rate for a single system could be limited. Hence, it is of interest to understand the controllability of systems using small sized codewords. In the extreme case, one can consider single bit codewords.

Given an integer k , a natural question is to find the optimal coding scheme with 2^k codes and a corresponding control law, u , that minimizes an asymptotic expected cost function

$$\lim_{n\to\infty} \mathrm{E}\left\|x_n\right\|^2, \tag{3}$$

where the expectation is taken with regard to the pdf, f_n. This is known to be a difficult problem in general. In this paper, we solve this problem for the case where $k = 1$ and when the resulting closed-loop system is a Markov transformation.

B. Optimal Finite Attention Bit Control Law

Represent the state x_n by the random variable X. Denote $M = 2^k$, then one can represent the control law, u, in terms of a partition $\{R_i\}_{i=1}^M$ of the real line as follows:

$$u(x) = \sum_{i=1}^{M} u^{(i)} \mathcal{X}_{R_i}(x), \tag{4}$$

where $\mathcal{X}_{R_i}(x)$ is the characteristic function:

$$\mathcal{X}_{R_i}(x) = \left\{ \begin{array}{ll} 1 & \text{if} \quad x \in R_i, \\ 0 & \text{otherwise}. \end{array} \right. \tag{5}$$

Hence, the cost in (3) can then be represented as

$$J_\infty = \lim_{n\to\infty} \sum_{i=1}^{M} \int_{R_i} (ax + u^{(i)})^2 f_n(x) \mathrm{d}x, \tag{6}$$

where f_n is the pdf of X. If the partition $\{R_i\}_{i=1}^M$ is fixed, then to minimize the one-step cost function $\mathrm{E}\left\|x_{n+1}\right\|^2$ is equivalent to minimizing $J = \sum_{i=1}^{M} \int_{R_i} (ax + u^{(i)})^2 f_n(x)\mathrm{d}x$ with respect to $u^{(i)}$ for each i. This in turn is equivalent to minimizing

$$\int_{R_i} (ax + u^{(i)})^2 f_n(x)\mathrm{d}x = P_i \int_{-\infty}^{\infty} (ax + u^{(i)})^2 f_{X|R_i}(x)\mathrm{d}x, \tag{7}$$

where $f_{X|R_i}(x)$ is the conditional pdf of X given that it is in the partition cell R_i , and P_i is the probability that X lies in R_i . The optimal value $u^{(i)}$ that minimizes this integral is [17]

$$u^{(i)} = -a \int_{R_i} x f_n(x)\mathrm{d}x / \int_{R_i} f_n(x)\mathrm{d}x. \tag{8}$$

In the following, we concentrate on feedback control law based on a 1-bit observation data. For notation simplicity, in this paper we consider controllers that are essentially symmetric around the origin. That is, controllers of the form:

$$u(x) = \left\{ \begin{array}{lll} -d & x & \geq & 0, \\ d & x & < & 0, \end{array} \right. \quad d > 0. \tag{9}$$

For the special case that the distribution f_n is symmetric around the origin, following from the previous discussion to optimize the one-step cost function, one should set the control value d as follows:

$$d = 2a \int_0^\infty x f_n(x)\mathrm{d}x \triangleq 2am_+(n). \tag{10}$$

We call this the one-step optimal controller. For such a controller, one can show that

$$\mathrm{E}\left\|x_{n+1}\right\|^2 = a^2(\mathrm{E}\left\|x_n\right\|^2 - 4m_+^2(n)). \tag{11}$$

Thus, the variance of the state is reduced if and only if $a^2(\mathrm{E}\left\|x_n\right\|^2 - 4m_+^2(n)) \leq \mathrm{E}\left\|x_n\right\|^2$. Hence, the value a must satisfy the inequality

$$a^2 \leq \mathrm{E}\left\|x_n\right\|^2 / (\mathrm{E}\left\|x_n\right\|^2 - 4m_+^2(n)). \tag{12}$$

C. Stability of the one attention bit Control Law

In the following, we investigate the stabilization effect of the proposed 1-bit feedback control law. The closed-loop system defined by such a controller can be represented by an expanding piecewise affine map

$$\tau(x) = \begin{cases} ax - d & x \geq 0, \\ ax + d & x < 0, \end{cases} \qquad a > 1. \tag{13}$$

Let τ^n denote the n-th iteration of τ, i.e., $\tau^n(x) = \tau \circ \tau \cdots \tau(x)$. Consider the orbit starting from the origin, $\{\tau^n(0)\}_{n\geq 1}$. If $d > d/(a-1)$, the orbits go to negative infinity, i.e., $\lim_{n\to\infty} \tau^n(0) \to -\infty$. Therefore, the following is a necessary condition for a system to be stabilizable:

$$a \leq 2. \tag{14}$$

An interval I is said to be invariant under τ , if $x_0 \in I$ implies $x_n \in I$ for all n. In the following Lemma, we show the existence of the invariant intervals for τ.

Lemma 1: For $1 < a \leq 2$, the whole real line is partitioned into the three τ-invariant intervals, i.e., $(-\infty, d/(a-1))$, $[-d/(a-1), d/(a-1)]$, and $(d/(a-1), +\infty)$. In addition, the interval $\hat{J} = [-d, d]$ and the points $-d/(a-1), d/(a-1)$ are invariant with respect to τ.

The proof of this result is straightforward and is omitted.

Denote the interval $[-d/(a-1), d/(a-1)]$ by I^* , $[-d, d]$ by $\hat{J}$ and the set $\hat{J} \cup \{-d/(a-1), d/(a-1)\}$ by $\tilde{J}$.

The concept of practical stability [2] is used to analyze the stabilization effect of coded feedback control laws. In this paper, we adopt the following related definition from [6].

Definition 1 [6]: Given two intervals $J \subseteq I$, which are τ-invariant, τ is said to be (I, J)-stable if for every $x_0 \in I$, there exists $n_0 > 0$ such that $x_n \in J$ for every $n \geq n_0$.

Let spt$f$ denote the support for the pdf of X. If the pdf of the initial state has a zero mean and sptf_0 is bounded, it is contained in a minimal closed interval $[-Y, Z]$. If the system starting from such an initial condition is stablizable, then Z should satisfy the inequality

$$aZ - d \leq Z. \tag{15}$$

which gives $Z \leq d/(a-1)$. Similarly, $Y \leq d/(a-1)$. This implies spt$f_0 \subseteq I^*$ for a stabilizable system. The following lemma provides further information on the convergence property of the map τ.

Lemma 2: For $1 < a \leq 2$, τ is $(I^*, \tilde{J})$-stable.

Proof: Let $x = d/(a-1) - \varepsilon$ with $0 < \varepsilon \leq d/(a-1)$. Then $\tau^n(x) = d/(a-1) - a^n\varepsilon$. Let $N = \lceil \log_a d/(a-1)\varepsilon \rceil$. It follows that $-d \leq \tau^{N+1}(x) = a\tau^N(x) - d \leq 0$ as $\tau^N(x) \geq 0$. Similarly, if $x = -d/(a-1) + \varepsilon$ $(0 < \varepsilon \leq d/(a-1))$, then $0 \leq \tau^{N+1}(x) \leq d$. Since $a \leq 2$, $ad - d \leq d$. This implies that for all $n \geq N$, the states $\tau_n(x) \in [-d, d]$ stay within the interval $\hat{J}$. Hence, starting from any point x in the open interval $(-d/(a-1), d/(a-1))$, $\tau^n(x)$ is in $\tilde{J}$ for all $n \geq N$. Since the points $\{-d/(a-1), d/(a-1)\}$ are invariant under τ, it follows that τ is $(I^*, \tilde{J})$-stable. □

III. INVARIANT MEASURES

From now on we assume that spt$f_0 \subseteq I^*$. To analyze the dynamics of the map (13) one can resort to the idea of an invariant measure. Let $L^1(I, \mathcal{B}, \lambda)$ be a normalized measure space with a Lebesgue measure λ on an interval I, $\mathcal{B}$ be the Borel field of I, and $\tau : I \to I$ be a measurable non-singular transformation. The Frobenius-Perron operator $P_\tau : L^1 \to L^1$ is defined as follows: for any $f \in L^1$, $P_\tau f$ is the unique function in L^1 such that [12]

$$\int_A P_\tau f \mathrm{d}\lambda = \int_{\tau^{-1}(A)} f \mathrm{d}\lambda \tag{16}$$

for any $A \in \mathcal{B}$. If X is a random variable taking value in I with a pdf f, the evolution of f under τ is then described in terms of the Frobenius-Perron operator as $P_\tau f$.

The measure μ, defined by $\mu(A) = \int_{\tau^{-1}(A)} f \mathrm{d}\lambda$, is said to be invariant with respect to the non-singular transformation $\tau : I \to I$ if

$$\mu(\tau^{-1}(A)) = \mu(A) \tag{17}$$

for all $A \in \mathcal{B}$. The invariant density f^* is a fixed point of P_τ, and f^* is τ-invariant if and only if

$$P_\tau f^* = f^* \tag{18}$$

almost everywhere. It is important to compute f^* since it describes the asymptotic properties of the orbit $\{\tau^n(x)\}_{n\geq 0}$.

Definition 2 [12]: Let $I = [b, c]$. The map $\tau : I \to I$ is called expanding piecewise monotonic if there exists a partition, $b = b_0 < b_1 < \cdots < b_q = c$, and $u \geq 1$ such that $\tau|_{[b_{i-1}, b_i]}$ is a C^u function and $|\tau'| > 1$ wherever the derivative exists.

For any expanding piecewise C^2 transformation, φ, it is proved in [10] that for any $f \in L^1$, the sequence $\sum_{k=0}^{n-1} P_\varphi^k f/n$ converges in norm to a function $f^* \in L^1$. Moreover, it has $P_\varphi f^* = f^*$.

Hence, there exists invariant measure for the expanding piecewise linear transformation τ defined in (13). It is shown that the Frobenius-Perron operator can be represented by a finite-dimensional matrix for piecewise linear Markov transformations, and the invariant density can be obtained from the induced matrix [14].

IV. INVARIANT DENSITIES OF PIECEWISE LINEAR MARKOV TRANSFORMATIONS

In this section, we focus on the analysis of the case where the map τ in (13) is a Markov transformation. The optimal control law that minimizes the asymptotic expected cost function is derived when the map in (13) is a covering.

A. Invariant densities of Markov Transformations

Let $\tau : I \to I$ be a piecewise linear map and let $\mathcal{P} = \{I_i\}_{i=1}^q$ be a partition of the interval $I = [b, c]$. Given the partition points $b = b_0 < b_1 < \cdots < b_q = c$, denote the restriction of τ to $I_i = (b_{i-1}, b_i)$ by τ_i. A partition $\mathcal{P}$ is a Markov partition under τ if every region I_i is mapped onto some connected union of intervals of $\mathcal{P}$ [15]. The corresponding map τ is called a Markov transformation.

A function that is piecewise constant over a partition of $\mathcal{P}$ can be represented by a row vector consisting of the corresponding constant values. Following the approach in [16], given a Markov partition $\mathcal{P}$ defined by τ and a function f that is piecewise constant over $\mathcal{P}$, the action of the Frobenius-Perron operator, P_τ, on f can be represented by a $q \times q$ matrix M_τ so that if $\pi^f = (\pi_1, \cdots, \pi_q)$ represents f, then $P_\tau f$ is represented by $\pi^f M_\tau$. The element of the matrix M_τ, $(m_{ij})_{1 \le i,j \le q}$, is of the form [16]

$$m_{ij} = a_{ij}/|\tau_i'| = \lambda(I_i \cap \tau^{-1}(I_i))/\lambda(I_i), 1 \le i, j \le q, \quad (19)$$

where λ denotes the Lebesgue measure. *Theorem 9.3.1* [16] shows that the induced matrix M_τ has the following properties: M_τ has 1 as the eigenvalue of maximum modulus; if M_τ is also irreducible, then the algebraic and geometric multiplicity of the eigenvalue 1 is also 1. The expanding Markov transformation has unique invariant measure when the matrix M_τ is irreducible [15]. Moreover, *Theorem 9.4.1* of [16] establishes that for an expanding piecewise linear Markov transformation, τ, every τ-invariant density function must be piecewise constant on the partition $\mathcal{P}$.

The following definition introduces the concept of asymptotical stability for a Frobenius-Perron operator [18].

Definition 3 [18]: Let $L^1(I, \mathcal{B}, \lambda)$ be a measure space and $P_\tau : L^1 \to L^1$ a Frobenius-Perron operator corresponding to a non-singular transformation $\tau : I \to I$. Then $\{P_\tau^n\}$ is said to be asymptotically stable if there exists a unique density function f^* such that $P_\tau f^* = f^*$ and $\lim_{n\to\infty} \|P_\tau^n f - f^*\| = 0$ for any $f \in L^1$. The transformation τ is then said to be statistically stable.

The following result is quite obvious and is needed for the proof of the main theorem.

Proposition 1: Let $L^1(I, \mathcal{B}, \lambda)$ be a measure space and let $\tau : I \to I$ be a non-singular map. For an integer $g \ge 1$, τ^g is statistically stable if and only if τ is statistically stable.

Proof: Assume τ is statistically stable. It implies that,

$$\lim_{n\to\infty} \|(P_\tau^g)^n f - f^*\| = 0$$

for any $f \in L^1$. Hence, τ^g is statistically stable.

Suppose τ^g is statistically stable. For an integer $n > g$, let $n = mg + l$, where m and l are positive integers. For any $f \in L^1$, it obtains,

$$\lim_{n\to\infty} \|P_\tau^n f - f^*\| = \lim_{n\to\infty} \|(P_\tau^g)^m (P_\tau^l f) - f^*\| = 0.$$

This implies that τ is statistically stable. $\square$

A mathematical concept called a covering is defined as follows: a map $\tau : I \to I$ is said to be a covering [6], if for any open interval $U \subseteq I$, there exists $t \in \mathbb{N}$ such that $\tau^t(U) = I$.

Theorem 1: Given the initial distribution f_0 with compact support, if the map τ in (13) is a covering, the optimal control value d^* that minimizes the cost function $J_\infty = \lim_{n\to\infty} E\|x_n\|^2$ is given by

$$d^* = \arg\min_d \; such\ that\ I^*(d) \supseteq \mathrm{spt} f_0.$$

Proof: Let $J_\infty(d)$ ($f_d^*(x)$) denote the cost function (invariant density) related to the control value d, respectively. Lemma 2 shows that τ is $(I^*, \tilde{J})$-stable as $1 < a \le 2$. Let $\mathcal{P} = \{I_i\}_{i=1}^q$ be a partition of the interval $\hat{J} = [-d, d]$ (the two isolated points $\{-d/(a-1), d/(a-1)\}$ can be ignored). Since τ is a covering, let m denote the smallest integer such that $\tau^m(I_i) = \hat{J}$, $i = 1, \ldots, q$. As shown in *Theorem 6.2.2.* [18], the map τ^m is statistically stable. By means of Proposition 1, the expanding map τ is statistically stable, i.e., $\lim_{n\to\infty} P_\tau^n f_0 \to f^*$.

It can be shown that the invariant density functions $f_1^*(x)$ and $f_d^*(x)$ are related as follows:

$$f_d^*(x) = (1/d) f_1^*(x/d), \quad x \in [-d, d].$$

Hence, we have

$$\begin{aligned} J_\infty(d) &= \lim_{n\to\infty} \int_{-d}^{d} x^2 P_\tau^n f_0 dx = \int_{-d}^{d} x^2 f_d^*(x) dx \\ &= \int_{-d}^{d} x^2 [(1/d) f_1^*(x/d)] dx = d^2 \int_{-1}^{1} x^2 f_1^*(x) dx. \quad (20) \end{aligned}$$

As a consequence,

$$J_\infty(d) = d^2 J_\infty(1). \quad (21)$$

This implies that the asymptotic cost function J_∞ can be minimized by minimizing d. That can be achieved by setting d^* as stated in the theorem. $\square$

Given there is a condition on τ being a covering, it is natural to ask how restrictive such a condition is. Proposition 2 partially addresses this question and is a slight modification of *Williams Theorem* [19] to adapt to the system considered here.

Proposition 2: If $a > \sqrt{2}$, the map τ in (13) is a covering over the interval $\hat{J}$.

Proof : Define an open interval $U \subseteq \hat{J}$. Using the proof of *Williams Theorem* [19], the interval $\tau^k(U)$ finally contains either $[0, d)$ or $(-d, 0]$. Assume $\tau^k(U) \supseteq [0, d)$. This implies

$$\begin{aligned} \tau^{k+1}(U) &\supseteq [-d, \tau(d)) = [-d, 0) \cup [0, \tau(d)) \\ \to \tau^{k+2}(U) &\supseteq [-d, \tau^2(d)) \cup [\tau(-d), d). \end{aligned}$$

The last union is equal to $[-d, d)$ if $\tau^2(d) - \tau(-d) = a^2 - 2 > 0$. This holds as $a > \sqrt{2}$. Similar arguments hold if $\tau^k(U) \supseteq (-d, 0]$. $\square$

Note that condition (iii) of *Williams Theorem* is not assumed in Proposition 2.

In the main theorem, the asymptotic stability of $\{P_\tau^n f\}$ is required to minimize the asymptotic expected cost function in (3). If the time-averaged cost function $J_\infty = \lim_{n\to\infty}\sum_{k=0}^{n-1}\|x_k\|^2/n$ is considered, the convergence result in [10] of the sequence $\sum_{k=0}^{n-1} P_\tau^k f_0/n$ to an invariant density function f^* can be applied and results similar to Theorem 1 can be obtained without the assumption that the map is a covering.

Computing the invariant density, f^*, appears to be a difficult task in general. In this paper, we focus on the computation of the invariant density when the map τ in (13) is a Markov transformation. It is hence interesting to investigate when τ is a Markov transformation. In [20], it was shown that, given the original partition $\mathcal{P} = \{[-d, 0), [0, d)\}$, $\mathcal{P}$ is a Markov partition of order r if and only if r is the smallest integer such that $\tau^r(\pm d) = 0$. A necessary and sufficient condition for τ to be a Markov transformation is also presented in terms of a polynomial [20] of the form

$$(-1)^m a^{s_m+1} - 2\sum_{i=1}^{m}(-1)^i a^{s_i} - 1 = 0, \qquad (22)$$

where $s_0 = 0$, $s_i = s_{i-1} + v_i$, $i = 1, \cdots, m$, and $\mathbf{v}$ is an integer vector $\mathbf{v}= [v_1, v_2, \cdots, v_m]$, with $v_i \le v_m$, such that m satisfies $\sum_{i=1}^m v_i = r$. It is conjectured in [20] that there are infinitely many solutions to (22).

We refine the conjecture here to state that there are infinitely many cases where the map τ in (13) is a Markov transformation even under the additional condition $1 < a < 2$. The conjecture has been verified in [21]. For lack of space, only an example is given here for illustration. Given an orbit $\{\tau^n(d)\}_{n\ge 0}$, define the sign vector as follows. Let $-(+)$ denote the sign of $\tau^i(x)$, where $-(+)$ represents strictly negative (strictly positive) values, respectively. The case $\tau^i(x) = 0$ is represented by 0. Let $\tau_a^i(d)$ denote the map related to a and d at the i-th iteration, where $1 < a < 2$ and $d > 0$. The points $\{\tau_{a_0}^i(d)\}_{i=0}^6$ have the sign vector of the form $\{+, +, +, -, +, -, 0\}$ when $a_0 \approx 1.75488$. For a value a_1, $a_1 > a_0$, there is an orbit with the sign vector of the form $\{+, +, +, -, +, -, +, -, 0\}$. Increasing the value a, it can be shown that there exist values of a_n, $\sqrt{2} < a_n < 2$, with sign vectors in the pattern of

$$\underbrace{\{+, +, +, -, +, -, +, -, \cdots, +, -, +, -, 0\}}_{2n}.$$

This implies that, for each a_n, $n \ge 3$, there exists a related Markov transformation.

When the map τ is not a Markov transformation there is no simple approach to solve the functional equation $P_\tau f^* = f^*$. In [13], an approach was proposed to approximate f^* by using the eigenvectors of certain matrices. However, the results in [13] require $|\tau'(x)| > 2$, and the computation of the matrices can be difficult when the operators are not Markov. A different method is presented in [14], which shows that an expanding piecewise linear map τ can be approximated by sequences of piecewise linear Markov transformations $\{\tau_n\}_{n\ge1}$, whose densities converge to the invariant density of τ. The approach to construct the piecewise linear Markov approximations sequence of τ can be found in [14].

B. Examples

Examples are given to illustrate the computation of the invariant density when the map in (13) is an expanding piecewise Markov transformation. Assume that the initial state is uniformly distributed in the interval $[-Z, Z]$:

$$f(x) = \begin{cases} 1/(2Z) & -Z \le x_0 \le Z, \\ 0 & others. \end{cases}$$

The one-step optimal 1-bit control law is obtained by setting $d = aZ/2$. The value of a satisfies $a \le 2$, which implies $Z \le d/(a-1)$ and ensures the closed-loop system is stabilizable. When $a = 2$, it follows that $d = Z$ and the invariant density is uniformly distributed in the interval $[-Z, Z]$. When $a < 2$, the trajectory $\{\tau^n([-Z, Z])\}_{n\ge0}$ converges to the interval $[-d, d]$. Consider the value $a = (1+\sqrt{5})/2$ (the specialized system parameter considered in [5]), which is a root of the equation $a^2 - a - 1 = 0$. Given the original partition points $\mathcal{Q}^{(0)} = \{-d, d, 0\}$, the partition points corresponding to τ and τ^2 are given by

$$\mathcal{Q}^{(1)} = \{-d, d, 0, -d/a, d/a\} \text{ and } \mathcal{Q}^{(2)} = \mathcal{Q}^{(1)}.$$

The Markov partition is thus obtained as the following

$$\mathcal{P} = \{[-d, -d/a), [-d/a, 0), [0, d/a), [d/a, d]\}.$$

The induced matrix M_τ associated with the partition is

$$M_\tau = \frac{1}{a}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Following the results in [16], the spectral radius of the primitive matrix M_τ is 1, and the eigenvector corresponding to the maximum eigenvalue 1 is given by

$$\pi^f = (0.3717, 0.6015, 0.6015, 0.3717)$$

The invariant density function can then be expressed as

$$f(x) = 0.3717\mathcal{X}_{I_1}(x) + 0.6015\mathcal{X}_{I_2}(x),$$

where $\mathcal{X}_{I_1}(x)$ is the characteristic function on the intervals $I_1 = [-d, -d/a) \cup (d/a, d]$ and $I_2 = [-d/a, d/a]$.

It is noted that a is a root of the polynomial (22), notice that $v_1 = 2$, $s_1 = 2$, $m = 1$ and $r = 2$. It follows that (22) can be rewritten as

$$a^3 - sa^2 + 1 = (a^2 - a - 1)(a - 1) = 0.$$

The root of this polynomial that is larger than 1 is given by $a = (1 + \sqrt{5})/2$.

For a second example, one can consider the equation $a^7 - a^6 - a^5 - a^4 + a^3 - a^2 - a + 1 = 0$ and one of its roots with the value $a_0 \approx 1.8071$. As **v**$= [1, 2, 1, 3]$, **s**$= [1, 3, 4, 7]$ and $r = 7$, it follows from (22), that

$$\begin{aligned} a^8 - 2a^7 + 2a^4 - 2a^3 + 2a - 1 &= \\ (a^7 - a^6 - a^5 - a^4 + a^3 - a^2 - a + 1)(a - 1) &= 0. \end{aligned}$$

Hence, a_0 is also a root of this polynomial.

As shown in Proposition 2, the map in (13) is a covering if $a > \sqrt{2}$. As a third example, consider a case where this map is not a covering. Particularly, consider the limiting value $a = \sqrt{2}$, which is a root of the equation $a^2 - 1 = 1$. Given the partition points $\mathcal{Q}^{(0)} = \{-d, -d/(a+1), d/(a+1), d, 0\}$, the Markov partition is obtained as

$$\{[-d, -\frac{d}{\sqrt{2}+1}), [\frac{d}{\sqrt{2}+1}, 0), [0, \frac{d}{\sqrt{2}+1}], (\frac{d}{\sqrt{2}+1}, d]\}.$$

The induced matrix M_τ associated with the partition is

$$M_\tau = \frac{1}{a}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The spectral radius of the reducible matrix M_τ is 1, and the eigenvector corresponding to the maximum eigenvalue 1 is

$$\pi^f = (0.4082, 0.5774, 0.5774, 0.4082).$$

The invariant density function can then be expressed as

$$f(x) = 0.4082\mathcal{X}_{I_1}(x) + 0.5774\mathcal{X}_{I_2}(x),$$

where $\mathcal{X}_{I_i}(x)$ is the characteristic function on the intervals

$$I_1 = [-d, -\tfrac{d}{a+1}) \cup (\tfrac{d}{a+1}, d] \text{ and } I_2 = [-\tfrac{d}{a+1}, \tfrac{d}{a+1}].$$

V. CONCLUSION

This paper presents results on the probabilistic dynamical behavior of systems under finite communication bandwidth feedback control. An essentially symmetric 1-bit control law is analyzed. The optimal control law that minimizes an asymptotic expected cost function is derived under the assumption that the closed-loop system is a covering. It is of interest to consider the multiple bit and multiple user situations. These extensions are currently being investigated.

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