SURE-BASED WAVELET THRESHOLDING INTEGRATING INTER-SCALE DEPENDENCIES

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Abstract

We propose here a new pointwise wavelet thresholding function that incorporates inter-scale dependencies. This non-linear function depends on a set of four linear parameters per subband which are set by minimizing Stein's unbiased MSE estimate (SURE). Our approach assumes additive Gaussian white noise.

In order for the inter-scale dependencies to be faithfully taken into account, we also develop a rigorous feature alignment processing, that is adapted to arbitrary wavelet filters (e.g. non-symmetric filters).

Finally, we demonstrate the efficiency of our denoising approach in simulations over a wide range of noise levels for a representative set of standard images.

Noisy signal model

- **In the image domain:** $\mathbf{g} = \mathbf{f} + \mathbf{w},\ w_n \sim \mathcal{N}(0,\sigma^2)$
- **After an orthonormal wavelet transform (OWT):** $\mathbf{y} = \mathbf{x} + \mathbf{b},\ b_n \sim \mathcal{N}(0,\sigma^2)$

$\mathbf{f}$: unknown noise-free image
$\mathbf{g}$: observed noisy image
$\mathbf{w}$: additive Gaussian white noise
σ^2: noise variance (assumed to be known)
$\mathbf{x}$: noise-free wavelet coefficients
$\mathbf{y}$: noisy wavelet coefficients
$\mathbf{b}$: wavelet coefficients of the additive Gaussian white noise

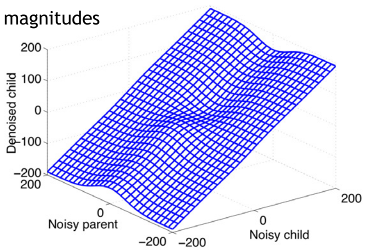
- **Consequence:** mean square error (MSE) preservation $\Rightarrow$ independent processing of the wavelet subbands

SURE-based wavelet thresholding

- **New** ***linearly*** **parameterized thresholding function which integrates the parent** Y**:**
  - Required properties:
    - Differentiable
    - Anti-symmetric
    - Linear for extreme magnitudes

$$\theta_0(y;\mathbf{u}) = \left(u_1 + u_2 \mathrm{e}^{-\frac{y^2}{12\sigma^2}}\right) y$$

$$\theta(y,Y;\mathbf{u},\mathbf{v}) = \underbrace{\mathrm{e}^{-\frac{Y^2}{12\sigma^2}}\theta_0(y;\mathbf{u})}_{\text{Low-SNR coefficients}} + \underbrace{\left(1-\mathrm{e}^{-\frac{Y^2}{12\sigma^2}}\right)\theta_0(y;\mathbf{v})}_{\text{High-SNR coefficients}}$$



- **SURE-based parameters optimization:**
  - Stein's unbiased MSE estimate (SURE):

Theorem: Given noisy $y_n = x_n + b_n$, with original unknown x_n and noise $b_n \sim \mathcal{N}(0,\sigma^2)$, the following random variable:

$$\epsilon = \frac{1}{N}\sum_{n=1}^{N}\left(\theta^2(y_n,Y_n;\mathbf{u},\mathbf{v}) - 2y_n\theta(y_n,Y_n;\mathbf{u},\mathbf{v}) + 2\sigma^2\frac{\partial\theta}{\partial y_n}(y_n,Y_n;\mathbf{u},\mathbf{v}) + y_n^2 - N\sigma^2\right)$$

is an unbiased estimate of the MSE, i.e. $\mathcal{E}\{\epsilon\} = \mathcal{E}\left\{\frac{1}{N}\sum_{n=1}^{N}\left|\theta(y_n,Y_n;\mathbf{u},\mathbf{v}) - x_n\right|^2\right\} = \mathcal{E}\{\text{MSE}\}$

  - Thanks to the **linear** parameterization and the **quadratic** form of ϵ in $(\mathbf{u},\mathbf{v})$:

SURE minimization $\Leftrightarrow$ resolution of a linear system of equations

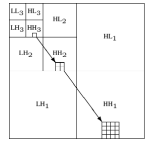
Aim: given $\theta(y,Y;\mathbf{p}) = \sum_{k=1}^{4} p_k f_k(y,Y)$, find the parameters $\mathbf{p} = \begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}$ which will minimize the MSE estimate ϵ:

$$\frac{\partial\epsilon}{\partial p_k} = 0,\ \forall k \in [1;4] \Longleftrightarrow \mathbf{p} = \mathbf{M}^{-1}\mathbf{c}$$

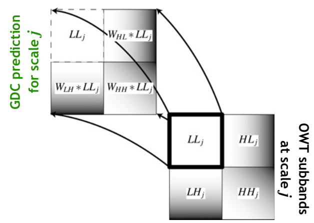
where: $m_{k,l} = \frac{1}{N}\sum_{n=1}^{N} f_k(y_n,Y_n) f_l(y_n,Y_n)$ and $c_k = \frac{1}{N}\sum_{n=1}^{N}\left(y_n f_k(y_n,Y_n) - \sigma^2 f_k'(y_n,Y_n)\right)$

Inter-scale dependencies in the OWT

- **Paradigm:** a coefficient at the current scale (child) depends on the coefficient at the next coarser scale (parent).
- **Usual approach:** expansion of the parent subband by a factor of 2.
  - **Limits:**
    - features misalignment
    - only the adjacent scale is taken into account



- **New approach:** construction of an inter-scale predictor Y out of the lowpass subband of the current scale
  1. Group delay compensation (GDC) with filter $W(z)$

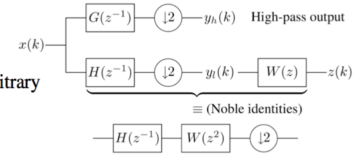


Theorem: In order for the output of a dyadic filterbank to be aligned, it is necessary and sufficient that:

$$W(z^2) = G(z^{-1})G(-z^{-1})(1+\lambda z^{-2})R(z^2)$$

where:

$$\begin{cases}\lambda = \pm 1\\ R(z) = R(z^{-1}) \text{ arbitrary}\end{cases}$$

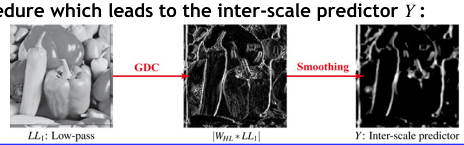


Required properties for W**:**
- Energy preservation
- Highpass behavior
- Shortest possible impulse response

Simple solution for symmetric filters: $W(z) = 1 - z$

  2. Homogenization by 2D-smoothing of the magnitude

Overview of the whole procedure which leads to the inter-scale predictor Y**:**



LL_j: Low-pass — $|W_{HL} * LL_j|$ — Y: Inter-scale predictor

Results and conclusions

- **PSNR comparison after 5 iterations of an OWT (*sym8*):**

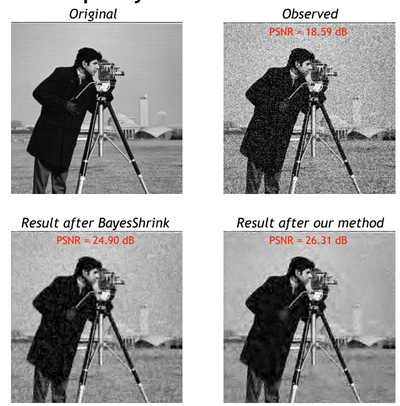
σ	5	10	20	30	50	100	5	10	20	30	50	100
Input PSNR	34.15	28.13	22.11	18.59	14.15	8.13	34.15	28.13	22.11	18.59	14.15	8.13
Method	Peppers 256 × 256						House 256 × 256					
BayesShrink	35.83	31.49	27.85	25.73	23.17	20.73	36.91	32.92	29.42	27.66	25.49	22.87
BiShrink 7 × 7	36.61	32.55	28.66	26.51	23.89	20.80	37.54	33.60	30.16	28.20	25.83	22.84
BLS-GSM 3 × 3	36.80	32.86	29.07	26.97	24.40	20.88	**38.01**	34.26	30.79	28.72	26.15	22.97
Our method	**37.17**	**33.18**	**29.33**	**27.13**	**24.43**	**21.32**	37.88	**34.29**	**30.93**	**28.98**	**26.58**	**23.51**
Method	Lena 512 × 512						Goldhill 512 × 512					
BayesShrink	37.01	33.44	30.24	28.54	26.50	23.91	35.93	31.94	28.69	27.13	25.41	23.32
BiShrink 7 × 7	37.56	34.23	31.00	29.15	26.87	23.91	36.17	32.27	29.07	27.44	25.57	23.26
BLS-GSM 3 × 3	37.84	**34.59**	31.32	29.41	27.07	24.06	36.37	32.61	29.41	27.73	25.73	23.30
Our method	**37.96**	34.55	**31.34**	**29.51**	**27.29**	**24.47**	**36.53**	**32.69**	**29.52**	**27.89**	**26.06**	**23.82**

- **Computation time:**
  - 0.6s for 256x256 images !
  - 2.7s for 512x512 images !

4 times faster than the best state-of-the-art with OWT !

- **Conclusions:**
  - New approach to image denoising with OWT without prior statistical modelization of the noise-free wavelet coefficients
  - Fast, robust and competitive SURE-based inter-scale wavelet thresholding algorithm

- **Visual quality:**



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