AN ALGEBRAIC OPTIMIZATION APPROACH TO IMAGE REGISTRATION

Zikai Sun, Zihan Zhang and Thierry Blu

Department of Electronic Engineering, The Chinese University of Hong Kong
{zksun, zhangzihan}@link.cuhk.edu.hk, thierry.blu@m4x.org

ABSTRACT

High-speed is an essential requirement for many applications of image registration. However, existing methods are usually time-consuming due to the difficulty of the task. In this paper, different from usual feature-based ideas, we convert the matching problem into an algebraic optimization task. By solving a series of quadratic optimization equations, the underlying deformation (rotation, scaling and shift) between image pairs can be retrieved. This process is extremely fast and can be performed in real-time. Experiments show that our method can achieve good performance at a much lower computation cost. When used to initialize our earlier parametric Local All-Pass (LAP) registration algorithm, the results obtained improve significantly over the state of the art.

***Index Terms*—** Image registration, Algebraic optimization, Local All-Pass filters.

1. INTRODUCTION

Image registration is the progress of estimating the geometric transformation between two images[22]. It is an essential topic in many fields, such as biomedical imaging, computer vision, remote sensing, and cartography. Although image registration has been studied for decades, still no algorithm is able to solve the problem consistently in all settings. In addition, many application scenarios of image registration put higher requirements on the calculation speed of the registration algorithm. For example, video stabilization requires real-time registration of high-definition images[17], Furthermore, remote sensing requires faster algorithms because satellite images are usually quite large.

The geometric distortion model is usually chosen to be, either global, typically involving parametric transformations with relatively few degrees of freedom (e.g., affine transformations), or local/elastic, typically involving transformations with many degrees of freedom. Elastic methods estimate the displacement field pixel-by-pixel and are suitable when the image pairs are related through a complex distortion. Various optimization criteria have been considered to address image registration[15]. Feature-based methods, for instance, estimate the displacement between the image pairs by identifying an optimal relationship between the extracted features of the image, such as minimum relative motion entropy [16], matching guided by locality preservation [9] and local linear transformations [10]. These approaches can reduce computational complexity, but the accuracy of feature extraction limits their effectiveness. Furthermore, there is even a risk that images without enough features cannot be registered.

Global registration methods establish a global parametric geometric relationship between images like, for instance, symmetric block-matching [13], DIRECT-type global optimization [21] and enhanced affine transformation [12]. However, these methods tend to use increasingly more parameters to represent more complex transformations. This complexity renders the computational load very heavy, and may not always lead to reliable results. Due to the need for speed in practical applications, we believe that a complete global model should be constructed using few parameters, so as to minimize the computation time.

To solve the problems mentioned above, we propose an algebraic optimization approach for global image registration(AR). More precisely, we express the image gradients using complex numbers (Wirtinger gradients, or Fourier-Argand gradients [20]) and solve several global optimization problems that will provide us, successively, with the rotation angle between the images, then their scaling factor and shift. It is worth noting that the optimization problems that we consider have closed-form solutions, hence that their implementation is very efficient. Furthermore, combining AR with our Local All-Pass (LAP)[4, 19] registration algorithm, can not only solve possible initialization issues, but also better take advantage of its ability to precisely estimate the displacement fields at the pixel level. The proposed method does not require any learning process and can be as fast as real-time image registration. The experimental results show that our method not only achieves highly accurate registration of images from global to local displacements but also runs in real-time.

2. METHOD

We first formulate the task using vector fields. Then, we design multiple optimization targets and estimate the similarity transformation according to the relationship between the optimal values. Finally, we combine it with the LAP[3] to obtain the elastic registration results.

2.1. Problem formulation

Consider two 2D continuous function $I_1(x)$ and $I_2(x)$ that represent the source and target image. We formulate the problem as finding a similarity transformation that characterised by $\mathrm{A} \in \mathbb{R}^{2\times 2}$ and $\mathrm{b} \in \mathbb{R}^2$, such that,

$$I_2(x) = I_1(\mathrm{A}x + \mathrm{b}) \tag{1}$$

Where $x = (x_1, x_2)^T \in \mathbb{R}^2$ is the pixel coordinates, b is the translation and $\mathrm{A} = s\times \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is the matrix containing rotation and scaling. We then express the gradient of the images as complex images (Wirtinger gradient)

$$g_1(z) = \nabla I_1^T \begin{bmatrix} 1 \\ i \end{bmatrix} \text{ and } g_2(z) = \nabla I_2^T \begin{bmatrix} 1 \\ i \end{bmatrix},$$

where $z = x_1 + ix_2 \in \mathbb{C}$. Note that we do not assume that, as a function of the complex variable z, these complex functions are analytic. When the images are noisy, we can alternatively use more robust gradients calculated from the Fourier-Argand representation [20]. The linear geometric transformation between the two images can also be expressed using complex variables, and their Wirtinger gradients satisfy the following relationship,

$$\begin{aligned} g_2(z_2) &= s^{-1}e^{-i\theta}g_1(z_1) \\ z_2 - z_0 &= s^{-1}e^{-i\theta}(z_1 - z_0) + t \\ \mathrm{d}^2 z_1 &= \mathrm{d}^2 z_2 \times s^2 \end{aligned} \tag{2}$$

Where g_2 is rotated by $\theta \in \mathbb{R}$ and scaling by $s \in \mathbb{R}$, around the center point $z_0 \in \mathbb{C}$, followed by translate $t \in \mathbb{C}$ pixels. d^2 is the differentiation notation over two dimension.

2.2. Optimization-based rotation acquisition

This subsection aims to estimate two images' rotation angles without knowing the scaling factor and translation relationship. So the estimation algorithm should be rotation-covariant under the premise of translation-invariant and scale-invariant. We propose to minimize a quadratic criterion $\mathcal{J}_g^0(w) = \int \mathfrak{Re}\{wg(z)\}^2 \mathrm{d}^2 z$ under the constraint that $|w| = 1$ for the vector field $g(z)$ defined in eq. (2). This is a constrained optimization problem format as follows,

$$\begin{aligned} &\min_{w\in\mathbb{C}} \quad \iint \mathfrak{Re}\{wg(z)\}^2 \mathrm{d}^2 z \\ &s.t. \quad |w|^2 = 1, \ \text{angl}(w) \in [-\frac{\pi}{2}, \frac{\pi}{2}) \end{aligned} \tag{3}$$

This optimization problem can be expressed in algebraic terms when the integral is discretized as a sum over all pixels. More specifically, since

$$2\mathfrak{Re}\{(wg(z)\} = wg(z) + \overline{wg(z)} \tag{4}$$

We have,

$$\begin{aligned} &\min_{\mathbf{x}} \quad ||\mathrm{Ax}||_F^2, \quad \mathrm{A} = [g(z), \overline{g(z)}] \in \mathbb{C}^{N\times 2} \\ &\text{s.t.} \quad ||\mathrm{Bx}||_F^2 = 1, \quad \mathrm{B} = \text{diag}(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), \quad x = [w, \bar{w}]^T \end{aligned} \tag{5}$$

Solve this equation by Lagrangian Multiplier, we have,

$$\mathcal{L} = \text{tr}(x^T \mathrm{A}^H \mathrm{A}x) - \text{tr}(\Lambda^T(x^T \mathrm{B}^H \mathrm{B}x - \mathrm{I})) \tag{6}$$

where $\Lambda \in \mathbb{R}^{d\times d}$ is the Lagrange multiplier matrix. Equating the derivative of $\mathcal{L}$ to zero gives:

$$\frac{\partial \mathcal{L}}{\partial x} = 2\mathrm{A}^H \mathrm{A}x - 2\mathrm{B}^T \mathrm{B}x\Lambda \overset{\text{set}}{=} 0 \tag{7}$$

This shows $A^H Ax = B^H Bx\Lambda$, Which is a generalized eigenvalue problem for $\tilde{\mathrm{A}} = \mathrm{A}^H \mathrm{A}$ and can be solved using singular value decomposition (SVD). This equation has two independent solutions, corresponding to two different real values of λ, The minimum of eq. (5) is obtained by choosing the smallest of these values. And the associated eigenvector should be multiplied by an adequate phase term $e^{i\alpha}$ so that, if $x = (w_1, w_2)$ we have $w_1 = \bar{w}_2$. The optimal value is finally obtained by,

$$w^* = \arg\min_w \mathcal{J}_g^0(w) = w_1 \tag{8}$$

Now let's consider two vector fields $g_1(z)$, $g_2(z)$ obtained from I_1 and I_2 respectively. If we have,

$$w_1^* = \arg\min_w \mathcal{J}_{g_1}^0(w), \ w_2^* = \arg\min_w \mathcal{J}_{g_2}^0(w) \tag{9}$$

then through the proof at the end of this subsection, we can get the optimal value to satisfy $w_1^{*2} = w_2^{*2}\mathrm{e}^{-2i\theta}$, which indicates the rotation angle as

$$\theta = \frac{i}{2}\ln(\frac{w_1^{*2}}{w_2^{*2}}) \tag{10}$$

Proof. Let's start with minimize the criterion for g_2, as follows

$$\begin{aligned} &\min_w \iint \mathfrak{Re}\{wg_2(z_2)\}^2 \mathrm{d}^2 z_2 \\ &= \min_w \iint \mathfrak{Re}\{ws^{-1}e^{-i\theta}g_1(z_1)\}^2 \mathrm{d}^2 z_1 \times s^{-2} \\ &= \min_{w'} \iint \mathfrak{Re}\{w'g_1(z_1)\}^2 \mathrm{d}^2 z_1 \end{aligned} \tag{11}$$

Where $w'^2 = w^2 e^{-i2\theta}$. Hence if we denote by w_1^*, w_2^* the parameter values for which $\mathcal{J}_{g_1}^0(w)$ and $\mathcal{J}_{g_2}^0(w)$ is minimum, then $w_1^2 = w_2^2\mathrm{e}^{-2i\theta}$. □

2.3. scale and shift acquisition by multiple optimizations

This section proposes a method to estimate the scale and shift, assuming the rotation angle is known. Together with section 2.2, a complete algebraic algorithm is formed to find the geometric similarities. To do so, we proposed to minimize several criteria successively, each minimization providing one equation. We first consider the following optimization,

$$\min_{w\in\mathbb{C}} \iint \mathfrak{Re}\{(w+az+b\bar{z})g(z)\}^2 \mathrm{d}^2 z \tag{12}$$

This is a quadratic optimization without constrains, we expand it to get,

$$\mathcal{L} = c_1 w^2 + c_2 \bar{w}^2 + 2c_3|w|^2 + 2c_4 w + 2c_5\bar{w} + c_6 \tag{13}$$

Where

$$\begin{aligned} c_1 = \bar{c}_2 &= \frac{1}{4}\iint g(z)^2 \mathrm{d}^2 z, \quad c_3 = \frac{1}{4}\iint |g(z)|^2 \mathrm{d}^2 z, \\ c_4 = \bar{c}_5 &= \frac{1}{4}\iint g(z)\left((az+b\bar{z})g(z) + (bz+a\bar{z})\overline{g(z)}\right)\mathrm{d}^2 z \end{aligned} \tag{14}$$

Calculate the derivative of $\mathcal{L}$ to w and $\bar{w}$, we have,

$$\frac{\partial \mathcal{L}}{\partial w} = \frac{\partial \mathcal{L}}{\partial \bar{w}} = c_1 w + c_3\bar{w} + c_4 \overset{\text{set}}{=} 0 \tag{15}$$

This system of equations has a solution as long as $\det = |c_1|^2 - c_3^2 \neq 0$. And the optimal value w^* can be obtained by solving this equation.

Here we introduce how to obtain s and t based on this carefully designed optimization. We rewrite z_2 in eq. (2) as,

$$z_2 = s^{-1}e^{-i\theta}z_1 + u \tag{16}$$

where $u = z_0(1-s^{-1}e^{-i\theta})+t$. This can be regarded as rotating and scaling the image by θ and s times, then shifting for u pixels. We denote eq. (12) by $\min_w \mathcal{J}_g(w,a,b)$, and solve following series of optimization equations: First, For some fixed a_1, b_1, if we have $w_1^* = \arg\min_w \mathcal{J}_{g_1}(w, a_1e^{-2i\theta}, b_1)$ and $w_2^* = \arg\min_w \mathcal{J}_{g_2}(w, a_1, b_1)$, then we can have following equation,

$$w_1^* = (w_2^* + a_1 u + b_1\bar{u})se^{-i\theta} \tag{17}$$

This can be proved briefly at the end of this subsection. Similarly, we consider minimize for some fixed a_2, b_2 that different from a_1, b_1. Let's assume $w_3^* = \arg\min_w \mathcal{J}_{g_1}(w, a_2e^{-2i\theta}, b_2)$, as well as $w_4^* = \arg\min_w \mathcal{J}_{g_2}(w, a_2, b_2)$, then we can get

$$w_3^* = (w_4^* + a_2 u + b_2\bar{u})se^{-i\theta} \tag{18}$$

Eventually, combining this redundant set of equations provides a possible expression for s and u. And θ is chosen to be one of the two values in eq. (10) that makes s and u as compatible as possible the equations eq. (17) and eq. (18). There are many options to choose parameters for acquiring s and u. And in practice, we set different parameters respectively to achieve this. This separable method will prevent inaccurate results due to estimating too many parameters at one time. A typical solution for estimate the shift is to use $a_1 = b_1 = 1, a_2 = -b_2 = 1$. And u can be calculated as,

$$\begin{aligned} \mathfrak{Re}\{u\} &= (w_1^* s^{-1}e^{-i\theta} - w_2^*)/2 \\ \mathfrak{Im}\{u\} &= (w_3^* s^{-1}e^{-i\theta} - w_4^*)/2 \end{aligned} \tag{19}$$

Proof. We start at optimizing the eq. (12) equation on g_2 and derive it by

$$\begin{aligned} &\min_w \iint \mathfrak{Re}[(w + az_2 + b\bar{z}_2)g_2(z_2)]^2 \mathrm{d}^2 z_2 \\ &= \min_w \iint \mathfrak{Re}[(w + a(s^{-1}e^{-i\theta}z_1 + u) \\ &\qquad + b(s^{-1}e^{i\theta}\bar{z}_1 + \bar{u}))s^{-1}e^{-i\theta}g_1(z_1)]^2 \mathrm{d}^2 z_1 \times s^{-2} \\ &= \min_{w'} \iint \mathfrak{Re}[(w' + a'z_1 + b\bar{z}_1)g_1(z_1)]^2 \mathrm{d}^2 z_1 \end{aligned} \tag{20}$$

Where $w' = (w + mu + n\bar{u})se^{-i\theta}$ and $a' = ae^{-2i\theta}$. This shows that if we have a package of optimization, where $a = a_1e^{-i2\theta}$ and $a = a_1$ separately, then eq. (17) and eq. (17) holds. □

2.4. Combination with Local All-Pass registration

LAP[4] is a fast and robust algorithm that treats image shifts in the spatial domain as phase changes in the frequency domain. The process is modeled by a local all pass filtering, written as $e^{-j\mathrm{u}^T\omega} = \hat{h}(\omega) = \frac{\hat{p}(e^{jw})}{p(e^{-jw})}$ Written in the spatial domain as forward and backward versions of filter p respectively, we have $p[-k] * I_2[k] = p[k] * I_1[k]$. Then, LAP approximates this filter by combining linear bases in the time domain, $p_{app}[k] = \sum_n c_n p_n[k]$. Finally, the displacement retrieved from the all pass filter is

$$u_{\text{LAP}}(x) = j\frac{\partial \log(H_{\text{app}}(e^{j\omega_1}, e^{j\omega_2}))}{\partial \omega_{1,2}}\bigg|_{\omega_1=\omega_2=0} \tag{21}$$

From here, we can see that this process may fail if the shift or rotation is larger than the filter size. This issue can be handled if we combine it with our Algebraic Registration algorithm while keeping the LAP's strengths for precise pixel-level estimation. Assuming the transform we obtained from AR is noted as A and b, then we have, $u_{\text{AR}}(x) = (\mathrm{I} - \mathrm{A}^{-1})x + \mathrm{A}^{-1}\mathrm{b}$ Together with the estimated displacement field between source image I_1 and affined target image $I_2(x)$. We have

$$u(x) = u_{\text{AR}}(x) + u_{\text{LAP}}(x - u_{\text{AR}}(x)) \tag{22}$$

And the final displacement field is

$$u(x) = (\mathrm{I} - \mathrm{A}^{-1})x + \mathrm{A}^{-1}\mathrm{b} + u_{\text{LAP}}(\mathrm{A}^{-1}x - \mathrm{A}^{-1}\mathrm{b}) \tag{23}$$

3. ANALYSIS

In this section, we analyze based on the following aspect: (1) The invariant property of the algorithm by different settings. (2) The speed of the algorithm.

3.1. Scale and translation invariance for the rotation estimation

One key reason our algorithm can succeed is its scaling and translation invariant when estimating the rotation angle. This

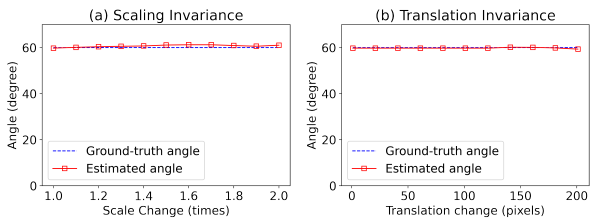

Fig. 1: Invariance property of the rotation estimation: When estimating the rotation angle between two images, the difference in image scale or translation have minor effect on the results.

Table 1: Running time of different algorithms.

Methods	SIFT flow[7]	NTG[2]	SelFlow[8]	**AR (Ours)**
Time (s)	84.7	15.5	4.3	**0.07**

section verifies the robustness of rotation estimation under various transformations. The experiments show in fig. 1. The results demonstrate that the scaling change and pixel translation accompanying the image rotation does not affect the rotation estimation.

3.2. Speed test

One of the advantages of our Algebraic Registration algorithm is its low complexity. This is because it only requires solving a series of optimization with closed-from solutions. We test our method on an image in 700×1000 pixels with an Intel Core i9-9880H. The speed comparison is shown in table 1. We can see that it is much faster than others.

4. EXPERIMENTS

4.1. Test on synthetic data

We numerically validate our method on synthetic images. For a raw image of 400×400 pixels, we first enlarge it to a size of 800 pixels using zero-padding. Then we construct the target image by rotating it by 60 degrees, scaling it by a factor of 1.5 and translating it by $(60, 40)$ pixels. We use the Median Absolute Error (E_{Med}), Average Absolute Error (E_{Mean}), the SalC[19] and running time as evaluation metrics. The results show in table 2. We can see that this method can have a good result with extremely fast speed for an image that LAP cannot handle. Combining AR with LAP, the method can be accurate with a relatively fast speed.

4.2. The oxford affine dataset

We compare our method with others in the Oxford Affine Dataset[11]. We use the Fourier-Argand gradient[20] in this test to replace the Wirtinger gradient. Since our results are even better than the provided displacement field(as shown in

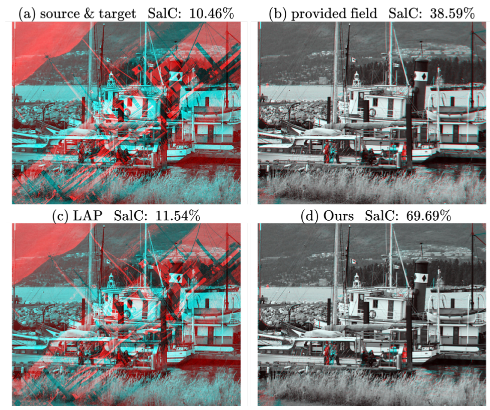

Fig. 2: Example registration results. (a). The source and target images are visualized in two colors. (b). Provided displacement field. (c). The original LAP method fails in this case. (d). Our method shows good performance.

Table 2: Performance test of synthetic images

	Synthethis Image (800 × 800 pixels)				
	E_{Med}	E_{mean}	Pars	SalC	Time (s)
LAP	7.15	7.66	0.979	41.66%	1.365
AR (ours)	1.47	1.41	0.976	81.09%	**0.079**
AR+LAP(ours)	**0.059**	**0.062**	0.976	**97.63%**	2.22

fig. 2 (b), which is not perfect), we evaluate algorithms by two reference-free metrics, called parsimony(Pars) and salience correlation(SalC)[19]. The results are shown in table 3. Our method can achieve better SalC results than others.

5. CONCLUSION

This paper proposes an algorithm that converts the image registration problem into an algebraic optimization problem. The method can estimate the geometric similarity of image pairs extremely fast, and can eliminate the deficiency of LAP at a low cost. Future works include real-time video stabilizers, group-wise registration, or high-resolution medical image registration.

Table 3: Methods comparison on the Oxford Affine Dataset

	Bikes		Trees		Leuven	
	Pars	SalC	Pars	SalC	Pars	SalC
Demons[18]	1.27	21%	1.28	28%	2.14	61%
MIRT[14]	3.08	44%	4.52	24%	3.45	59%
bUnwarpJ[1]	0.22	51%	0.93	30%	0.67	66%
SIFT flow[7]	0.71	52%	3.08	40%	0.92	64%
Elastix[5]	2.44	36%	2.44	28%	0.85	66%
SelFlow[8]	0.64	49%	1.41	32%	0.44	64%
LAP[4]	0.21	52%	0.77	35%	0.28	67%
LAFP[6]	0.23	42%	0.64	26%	0.29	51%
paraLAP[19]	0.18	53%	0.58	33%	0.15	67%
NTG[2]	0.22	52%	0.58	28%	0.11	61%
AR+LAP (Ours)	0.26	**71%**	0.70	**43%**	0.25	**74%**

References

[1] Ignacio Arganda-Carreras, Carlos OS Sorzano, Roberto Marabini, José María Carazo, Carlos Ortiz-de Solorzano, and Jan Kybic. Consistent and elastic registration of histological sections using vector-spline regularization. In *International Workshop on Computer Vision Approaches to Medical Image Analysis*, pages 85–95. Springer, 2006.

[2] Shu-Jie Chen, Hui-Liang Shen, Chunguang Li, and John H Xin. Normalized total gradient: A new measure for multispectral image registration. *IEEE Transactions on Image Processing*, 27(3):1297–1310, 2017.

[3] Christopher Gilliam and Thierry Blu. Local all-pass filters for optical flow estimation. In *2015 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pages 1533–1537. IEEE, 2015.

[4] Christopher Gilliam and Thierry Blu. Local all-pass geometric deformations. *IEEE Transactions on Image Processing*, 27(2):1010–1025, 2017.

[5] Stefan Klein, Marius Staring, Keelin Murphy, Max A Viergever, and Josien PW Pluim. Elastix: a toolbox for intensity-based medical image registration. *IEEE transactions on medical imaging*, 29(1):196–205, 2009.

[6] Wen-Yan Lin, Linlin Liu, Yasuyuki Matsushita, Kok-Lim Low, and Siying Liu. Aligning images in the wild. In *2012 IEEE Conference on Computer Vision and Pattern Recognition*, pages 1–8. IEEE, 2012.

[7] Ce Liu, Jenny Yuen, and Antonio Torralba. Sift flow: Dense correspondence across scenes and its applications. *IEEE transactions on pattern analysis and machine intelligence*, 33(5):978–994, 2010.

[8] Pengpeng Liu, Michael Lyu, Irwin King, and Jia Xu. Selflow: Self-supervised learning of optical flow. In *Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition*, pages 4571–4580, 2019.

[9] Jiayi Ma, Junjun Jiang, Huabing Zhou, Ji Zhao, and Xiaojie Guo. Guided locality preserving feature matching for remote sensing image registration. *IEEE transactions on geoscience and remote sensing*, 56(8):4435–4447, 2018.

[10] Jiayi Ma, Huabing Zhou, Ji Zhao, Yuan Gao, Junjun Jiang, and Jinwen Tian. Robust feature matching for remote sensing image registration via locally linear transforming. *IEEE Transactions on Geoscience and Remote Sensing*, 53(12):6469–6481, 2015.

[11] Krystian Mikolajczyk, Tinne Tuytelaars, Cordelia Schmid, Andrew Zisserman, Jiri Matas, Frederik Schaffalitzky, Timor Kadir, and Luc Van Gool. A comparison of affine region detectors. *International journal of computer vision*, 65(1):43–72, 2005.

[12] Chaobo Min, Yan Gu, Yingjie Li, and Feng Yang. Non-rigid infrared and visible image registration by enhanced affine transformation. *Pattern Recognition*, 106:107377, 2020.

[13] Marc Modat, David M Cash, Pankaj Daga, Gavin P Winston, John S Duncan, and Sébastien Ourselin. Global image registration using a symmetric block-matching approach. *Journal of medical imaging*, 1(2):024003, 2014.

[14] Andriy Myronenko and Xubo Song. Intensity-based image registration by minimizing residual complexity. *IEEE transactions on medical imaging*, 29(11):1882–1891, 2010.

[15] Sayan Nag. Image registration techniques: a survey. *arXiv preprint arXiv:1712.07540*, 2017.

[16] Feng Shao, Zhaoxia Liu, and Jubai An. Feature matching based on minimum relative motion entropy for image registration. *IEEE Transactions on Geoscience and Remote Sensing*, 2021.

[17] Fuhao Shi, Sung-Fang Tsai, Youyou Wang, and Chia-Kai Liang. Steadiface: Real-time face-centric stabilization on mobile phones. In *2019 IEEE International Conference on Image Processing (ICIP)*, pages 4599–4603. IEEE, 2019.

[18] Tom Vercauteren, Xavier Pennec, Aymeric Perchant, and Nicholas Ayache. Diffeomorphic demons: Efficient non-parametric image registration. *NeuroImage*, 45(1):S61–S72, 2009.

[19] Xinxin Zhang, Christopher Gilliam, and Thierry Blu. All-pass parametric image registration. *IEEE Transactions on Image Processing*, 29:5625–5640, 2020.

[20] Tianle Zhao and Thierry Blu. The fourier-argand representation: An optimal basis of steerable patterns. *IEEE Transactions on Image Processing*, 29:6357–6371, 2020.

[21] Cuicui Zheng, James Calvin, and Craig Gotsman. A direct-type global optimization algorithm for image registration. *Journal of Global Optimization*, 79(2):431–445, 2021.

[22] Barbara Zitova and Jan Flusser. Image registration methods: a survey. *Image and vision computing*, 21(11):977–1000, 2003.