D-SLAM: DIFFUSION SOURCE LOCALIZATION AND TRAJECTORY MAPPING

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ABSTRACT

We consider physical fields induced by a finite number of instantaneous diffusion sources, which we sample using a mobile sensor, along unknown trajectories composed of multiple linear segments. We address the problem of estimating the sources, as well as the trajectory of the mobile sensor. Within this framework, we propose a method for localizing sources of unknown amplitudes, and known activation times. The reconstruction method we propose maps the measurements obtained using the mobile sensor to a sequence of generalized field samples. From these generalized samples, we can then retrieve the locations of the sources as well as the trajectory of the sensor (up to a linear geometric transformation).

Index Terms— Diffusion fields, finite rate of innovation (FRI), field and trajectory reconstruction, sampling theory.

1. INTRODUCTION

Estimation of physical fields from sensors' measurements has attracted a lot of interest recently. We refer for example to the recent papers [1–5]. With the exception of [5], these papers assume sensor locations are known. Moreover sensors are at fixed locations. Sampling physical fields along trajectories using mobile sensors was considered in [6, 7], but in both cases the trajectories are known.

In this paper, we consider the problem of estimating the physical field of a finite number of instantaneous diffusion sources, from samples taken along *unknown* trajectories. We assume that the sampling trajectory consists of multiple linear segments, and that the activation times of the sources are known. Moreover, we leverage the assumption that the observation window is sufficiently short, such that the fluctuations in the observed field are negligible.

Some of the real-life applications of efficient sampling and reconstruction of such fields are the following. Consider for instance, the retracing of nuclear or chemical leakages [8], or detecting flames and smoke generated by forest fires. In these environmental monitoring applications, the use of unmanned aerial vehicles (UAVs) such as drones, is becoming increasingly popular. These vehicles are typically equipped with GPS sensors in order to track their position, which may not accurately work in inaccessible areas. It is therefore essential to develop algorithms that would allow the vehicle to estimate its position in space, without relying on GPS, whilst at the same time being able to estimate the physical field being monitored.

In order to develop an algorithm for estimation of the field sources as well as the sampling trajectory, we leverage the method in [9]. Here, the authors consider the problem of estimating a physical sampling field, e.g. an image, using samples from a moving sensor whose location is unknown. The algorithm proposed in [9] achieves reconstruction of both the sampling trajectory (up to a linear transformation and a shift), as well as the 2D image being sampled.

In this paper, we extend this method to the case of instantaneous diffusion sources, and show that we can retrieve the sampling trajectory composed of linear segments, as well as the locations of the diffusion sources, up to a linear geometric transformation.

2. PROBLEM FORMULATION

Let us consider the diffusion field induced by an instantaneous source (localized in both space and time), within a two-dimensional region. The diffusion field will propagate according to the Green's function [1], as follows:

$$f(\mathbf{x}, t) = \frac{1}{4\pi\mu(t-\tau_k)} a_k e^{-\frac{|\mathbf{x}-\mathbf{S}_k|^2}{4\mu(t-\tau_k)}} H(t-\tau_k), \qquad (1)$$

where:

a_k = amplitude of the diffusion source,
τ_k = activation time of the diffusion source,
$\mathbf{S}_k$ = coordinates of the source in $\mathbb{R}_2$ space,
$H(t)$ = unit step function,
μ = diffusivity of the medium.

Furthermore, suppose that within the time interval of observation, the evolution of the diffusion field is imperceptible, and that we know the activation time τ_k of the source. Under these assumptions, the reconstruction problem is equivalent to retrieving the location $\mathbf{S_k}$ and intensity a_k, from the spatial measurements:

$$f(\mathbf{x}) = \frac{1}{\pi C} a_k e^{-\frac{||\mathbf{x}-\mathbf{S}_k||^2}{C}},$$

where C is a constant equal to $C = 4\mu(t - \tau_k)$, and t is the start time of the observation.

For simplicity, we assume $C = 1$. Then, suppose we sample a field induced by K diffusion sources, and take measurements along a trajectory made up of L lines. Assume also that the measurements are taken at uniform points, with a step size equal to T, and that the length of each line is known and equal to l_j. Let us denote the start point of each line j with $\mathbf{b}_j \in \mathbb{R}_2$ (see Fig. 1), and describe the equation of this line as follows:

$$\mathbf{P}_j = \mathbf{b}_j + y(\mathbf{b}_{j+1} - \mathbf{b}_j), \tag{2}$$

where $y \in [0, 1]$.

Then, the measurements along line j, due to a number K of sources with coordinates $\mathbf{S}_k$ are given by:

$$f(\mathbf{P}_j) = \frac{1}{\pi} \sum_{k=1}^{K} a_k e^{-||\mathbf{P}_j - \mathbf{S}_k||^2}, \tag{3}$$

where we can express $||\mathbf{P}_j - \mathbf{S}_k||^2$ as follows:

$$\begin{aligned} ||\mathbf{P}_j - \mathbf{S}_k||^2 &= ||\mathbf{b}_j - \mathbf{S}_k + y(\mathbf{b}_{j+1} - \mathbf{b}_j)||^2 \\ &= ||\mathbf{b}_j - \mathbf{S}_k||^2 + 2y(\mathbf{b}_j - \mathbf{S}_k)^T(\mathbf{b}_{j+1} - \mathbf{b}_j) \\ &+ y^2||(\mathbf{b}_{j+1} - \mathbf{b}_j)||^2 \\ &\overset{(a)}{=} (l_j y + \frac{(\mathbf{b}_j - \mathbf{S}_k)^T(\mathbf{b}_{j+1} - \mathbf{b}_j)}{l_j})^2 \\ &+ ||\mathbf{b}_j - \mathbf{S}_k||^2 - (\frac{(\mathbf{b}_j - \mathbf{S}_k)^T(\mathbf{b}_{j+1} - \mathbf{b}_j)}{l_j})^2, \end{aligned} \tag{4}$$

where (a) follows from the assumption that the length of segment j is $||(\mathbf{b}_{j+1} - \mathbf{b}_j)|| = l_j$.

Therefore, the cumulative measurements along line j from all sources given in Eq. (3), become:

$$\begin{aligned} f(\mathbf{P}_j) = f_j(y) &= \frac{1}{\pi} \sum_{k=1}^{K} a_k e^{-||\mathbf{P}_j - \mathbf{S}_k||^2} \\ &= \frac{1}{\pi} \sum_{k=1}^{K} a_k e^{-||\mathbf{b}_j - \mathbf{S}_k||^2 + (\frac{(\mathbf{b}_j - \mathbf{S}_k)^T(\mathbf{b}_{j+1} - \mathbf{b}_j)}{l_j})^2} \\ &\times e^{-(l_j y + \frac{(\mathbf{b}_j - \mathbf{S}_k)^T(\mathbf{b}_{j+1} - \mathbf{b}_j)}{l_j})^2} \\ &:= \sum_{k=1}^{K} A_{k,j} e^{-(l_j y - Y_{k,j})^2}, \end{aligned} \tag{5}$$

where:

$$\begin{aligned} A_{k,j} &= a_k e^{-||\mathbf{b}_j - \mathbf{S}_k||^2 + (\frac{(\mathbf{b}_j - \mathbf{S}_k)^T(\mathbf{b}_{j+1} - \mathbf{b}_j)}{l_j})^2}, \\ Y_{k,j} &= \frac{(\mathbf{S}_k - \mathbf{b}_j)^T(\mathbf{b}_{j+1} - \mathbf{b}_j)}{l_j}. \end{aligned} \tag{6}$$

At discrete uniform points $y = nT$ with $n \in \mathbb{N}$, this becomes:

$$f_j(nT) = \sum_{k=1}^{K} A_{k,j} e^{-(l_j nT - Y_{k,j})^2}. \tag{7}$$

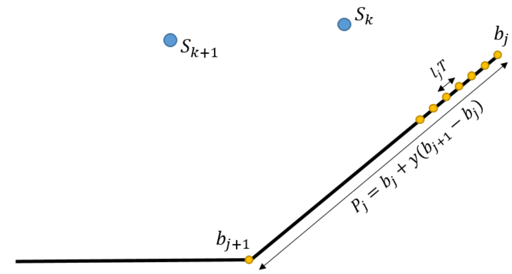

Fig. 1: Measurements at discrete time along linear trajectories, of the physical field created by two diffusion sources.

In order to localize the sources and estimate the trajectory of the mobile sensor, we need to first estimate the parameters $Y_{k,j}$ and $A_{k,j}$ from the measurements in Eq. (7), and one method to solve this problem is presented in Appendix A.

3. SIMULTANEOUS SOURCE LOCALIZATION AND MAPPING

Let us assume that the diffusion field is induced by $K \geq 3$ sources of *unknown* amplitudes, and that the trajectory is composed of $L \geq 2$ lines. Under this assumption, we will show that the locations of the sources and the lines that form the sampling trajectory can be correctly estimated, up to a geometric rotation and reflection. We first estimate the lines of the trajectory. Then, we recover the locations of the instantaneous sources, by ensuring these are consistent with the measurements we obtain. This guarantees that the locations of the sources are correct, relative to the estimated trajectory.

3.1. Frequency Pairing

In order to be able to use the parameters $Y_{k,j}$ (estimated as described in Section 2) for source localization, we need to first ensure these are paired across different lines. In other words, we need to identify whether two parameters $Y_{k,j}$ and $Y_{k',i}$ estimated using measurements on line j and i respectively, correspond to the same source $\mathbf{S}_k$, i.e. whether $k = k'$.

Using the derivations in Eq. (6), we get:

$$\log\left(\frac{\pi A_{k,j}}{a_k}\right) = Y_{k,j}^2 - ||\mathbf{b}_j - \mathbf{S}_k||^2, \tag{8}$$

which gives:

$$\begin{aligned} \log\left(\frac{A_{k,j}}{A_{k,j+1}}\right) &= Y_{k,j}^2 - Y_{k,j+1}^2 + ||\mathbf{b}_{j+1} - \mathbf{S}_k||^2 - ||\mathbf{b}_j - \mathbf{S}_k||^2 \\ &\overset{(b)}{=} Y_{k,j}^2 - Y_{k,j+1}^2 - 2l_j Y_{k,j} + l_j^2 = (Y_{k,j} - l)^2 - Y_{k,j+1}^2, \end{aligned}$$

where (b) follows from:

$$\begin{aligned} -2l_j Y_{k,j} &= ||\mathbf{b}_j - \mathbf{S}_k + \mathbf{b}_{j+1} - \mathbf{b}_j||^2 \\ &- ||\mathbf{b}_j - \mathbf{S}_k||^2 - ||\mathbf{b}_{j+1} - \mathbf{b}_j||^2 \\ &= ||\mathbf{b}_{j+1} - \mathbf{S}_k||^2 - ||\mathbf{b}_j - \mathbf{S}_k||^2 - l_j^2 \end{aligned}$$

In general, we match two frequencies $Y_{k,j}$ and $Y_{k',i}$ with corresponding amplitudes $A_{k,j}$ and $A_{k',j}$ respectively, if $\log\left(\frac{A_{k,i}}{A_{k',j}}\right) = (Y_{k,i} - l_j)^2 - Y_{k',j}^2$. If this holds, then $k = k'$.

3.2. Estimation of the Trajectory of the Mobile Sensor

Using the method in Section 2, we obtain the parameters:

$$
\begin{aligned}
Y_{k,j} &= \frac{(\mathbf{S}_k - \mathbf{b}_j)^T(\mathbf{b}_{j+1} - \mathbf{b}_j)}{l_j} \\
&= (\frac{\mathbf{b}_{j+1} - \mathbf{b}_j}{l_j})^T(\mathbf{S}_k - \mathbf{b}_j) := \mathbf{c}_j^T(\mathbf{S}_k - \mathbf{b}_j).
\end{aligned} \tag{9}
$$

Using the parameters $Y_{k,j} = \mathbf{c}_j^T(\mathbf{S}_k - \mathbf{b}_j)$, we can also obtain the difference $\Omega_{j,q} = Y_{k,j} - Y_{q,j}$ for each line j and any two different sources k and q:

$$
\Omega = \begin{bmatrix} \mathbf{c}_1^T(\mathbf{S}_2 - \mathbf{S}_1) & \mathbf{c}_1^T(\mathbf{S}_3 - \mathbf{S}_2) & \dots & \mathbf{c}_1^T(\mathbf{S}_1 - \mathbf{S}_K) \\ \mathbf{c}_2^T(\mathbf{S}_2 - \mathbf{S}_1) & \mathbf{c}_2^T(\mathbf{S}_3 - \mathbf{S}_2) & \dots & \mathbf{c}_2^T(\mathbf{S}_1 - \mathbf{S}_K) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{c}_L^T(\mathbf{S}_2 - \mathbf{S}_1) & \mathbf{c_L}^T(\mathbf{S}_3 - \mathbf{S}_2) & \dots & \mathbf{c}_L^T(\mathbf{S}_1 - \mathbf{S}_K) \end{bmatrix}
$$

$$
= \underbrace{\begin{bmatrix} \mathbf{c}_1^T \\ \mathbf{c}_2^T \\ \vdots \\ \mathbf{c}_L^T \end{bmatrix}}_{L\times 2} \underbrace{\begin{bmatrix} \mathbf{S}_2 - \mathbf{S}_1 & \mathbf{S}_3 - \mathbf{S}_2 & \dots & \mathbf{S}_1 - \mathbf{S}_K \end{bmatrix}}_{2\times K} := \mathbf{CS}
$$

The rank of matrix $\mathbf{\Omega}$ satisfies rank$(\mathbf{\Omega}) \leq 2$. This means that we need $K \geq 3$ sources and $L \geq 2$ to retrieve the matrix of line parameters $\mathbf{C}$, and the matrix of distances between sources $\mathbf{S}$, up to a linear transformation $\mathbf{Q}$: $\tilde{C} = C\mathbf{Q}^{-1}$ and $\tilde{\mathbf{S}} = \mathbf{QS}$.

Since parameter $\mathbf{c}_j$ of any line j satisfies $||\mathbf{c}_j|| = 1$, we normalize the rows of the estimated matrix $\tilde{C}$. Once we find the parameters $\tilde{\mathbf{c}}_j$, we can then sequentially retrieve the start points of all lines using $\mathbf{b}_{j+1} = \mathbf{b}_j + l_j\tilde{\mathbf{c}}_j$, for $j \geq 1$, where $\mathbf{b}_1$ is arbitrarily set, and l_j is the length of each line j.

The linear transformation $\mathbf{Q}$ can be estimated from the equations of two lines i and j, by solving:

$$
\begin{bmatrix} \tilde{\mathbf{c}}_i^T \\ \tilde{\mathbf{c}}_j^T \end{bmatrix} \mathbf{Q} = \begin{bmatrix} \mathbf{c}_i^T \\ \mathbf{c}_j^T \end{bmatrix}
$$

In other words, if two lines of the trajectory are known, we can perfectly retrieve the parameters of all other lines.

3.3. Localization of Sources

We leverage the results in Section 2 to retrieve the parameters $Y_{k,j}$, corresponding to source $\mathbf{S}_k$ and line j. Moreover, we arbitrarily set $\mathbf{b}_1$ and $\mathbf{c}_1$ and retrieve the parameters $\mathbf{b}_j$ and $\mathbf{c}_j$ of all other lines as in Section 3.2.

Then, the location of the source $\mathbf{S}_k$ can be retrieved using the estimated parameters $Y_{k,i}$ and $Y_{k,j}$, corresponding to lines i and j respectively, as follows.

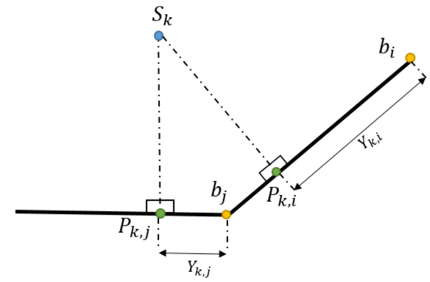

Fig. 2: Estimating the location of source $\mathbf{S}_k$, using measurements along two different lines i and j.

First, the parameter $Y_{k,j} = \frac{(\mathbf{S}_k - \mathbf{b}_j)^T(\mathbf{b}_{j+1} - \mathbf{b}_j)}{l_j}$ represents the scalar projection of vector $(\mathbf{S}_k - \mathbf{b}_j)$ onto vector $(\mathbf{b}_{j+1} - \mathbf{b}_j)$. Then, the vector projection $(\mathbf{P}_{k,j} - \mathbf{b}_j)$ of $(\mathbf{S}_k - \mathbf{b}_j)$ onto vector $(\mathbf{b}_{j+1} - \mathbf{b}_j)$ can be computed as:

$$
\mathbf{P}_{k,j} - \mathbf{b}_j = Y_{k,j}\frac{\mathbf{b}_{j+1} - \mathbf{b}_j}{l_j}, \tag{10}
$$

where $\mathbf{P}_{k,j}$ is the point where the perpendicular from $\mathbf{S}_k$ to line j, intersects line j, as seen in Fig. 2.

Hence, the inner product between vectors $\mathbf{P}_{k,j} - \mathbf{b}_j$ and $\mathbf{S}_k - \mathbf{P}_{k,j}$ must be zero:

$$
(\mathbf{P}_{k,j} - \mathbf{b}_j)^T \cdot (\mathbf{S}_k - \mathbf{P}_{k,j}) = 0. \tag{11}
$$

Similarly, for segment i, we obtain:

$$
(\mathbf{P}_{k,i} - \mathbf{b}_i)^T \cdot (\mathbf{S}_k - \mathbf{P}_{k,i}) = 0. \tag{12}
$$

Since the source $\mathbf{S}_k$ is located both on the perpendicular to line i, as well as on the perpendicular to line j, its coordinates must satisfy both Eq. (11) and (12), as depicted in Fig. 2. We can therefore uniquely retrieve the location of any source $\mathbf{S}_k$, by solving the system of Eq. (11) and (12).

Finally, once the location $\mathbf{S}_k$ has been estimated, we can compute the intensity of the source from the parameters $A_{k,j}$, retrieved using the method in Section 2.

4. EXPERIMENTAL RESULTS

We consider the case of three diffusion sources of *unknown* amplitudes, whose field we sample along multiple lines. We estimate the sources and the sampling trajectory using the method presented in Section 3. The results in Fig. 3 show that we can retrieve the locations of the sources and the lines up to a linear transformation (translation, reflection, orthogonal rotation, and scaling), when we arbitrarily set the start point of the trajectory. The results in Fig. 4 show that we can exactly retrieve the locations of the sources, when the first two segments of the trajectory are known.

In many practical situations, the field measurements may be corrupted by noise, and this may lead to errors in the parameters $Y_{k,j}$ and $A_{k,j}$ estimated using the method in Appendix A. In this case, we can use Procrustes analysis [10]

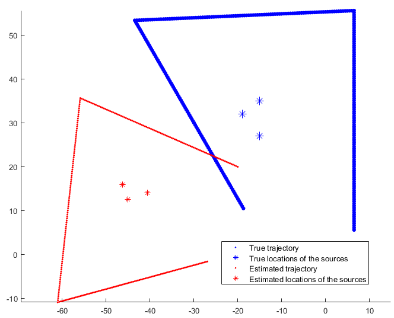

Fig. 3: Estimation of the locations of 3 sources and the sampling trajectory composed of 3 linear segments, when the starting point of the trajectory is arbitrary.

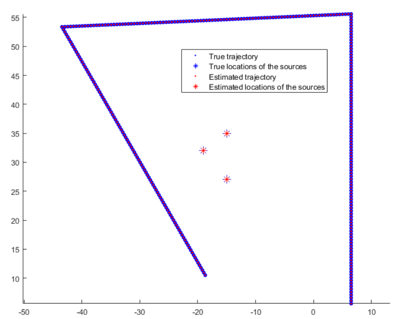

Fig. 4: Estimation of the locations of 3 sources and the sampling trajectory composed of 3 linear segments, when the first two segments of the trajectory are known.

to estimate a linear transformaton between the estimated and actual trajectories and source locations respectively. For example, for the setting depicted in Fig. 5, when the field measurements are corrupted by white, additive Gaussian noise of SNR$= 20dB$, the average squared error between the linearly transformed points of the estimated trajectory and those of the true trajectory is $d_L = 0.7144$, whereas the average error of the source locations is $d_S = 4.37 \times 10^{-4}$.

We can further account for possible inaccuracies in the estimated parameters $Y_{k,j}$ and $A_{k,j}$ as follows. First, we can increase the number L of lines in the trajectory, and choose the location of each source $\tilde{\mathbf{S}}_k$ which agrees with most of the estimated parameters $Y_{k,j}$, for $j = 1, 2, ..., L$, using the method described in Section 3.3. In addition, we could adjust the estimated location $\tilde{\mathbf{S}}_k$ and amplitude $\tilde{a}_k$ of each source such that the estimated measurements $\frac{1}{\pi}\sum_{k=1}^{K} \tilde{a}_k e^{-||\mathbf{P}_j - \tilde{\mathbf{S}}_k||^2}$ agree to the actual measurements in Eq. (7), along all lines $j = 1, 2, ..., L$.

5. CONCLUSIONS

In this paper, we have presented an algorithm for estimating the locations of multiple diffusion sources, from samples taken along unknown trajectories. When the activation times of the sources are known, the algorithm retrieves their locations, as well as the sampling trajectory, up to a linear transformation. Simulations performed on synthetic data validate the proposed reconstruction method. In future work, we would like to understand the behaviour of the algorithm for source localization and trajectory mapping under noisy conditions. We also hope to extend this to the case when the activation times of the sources are unknown.

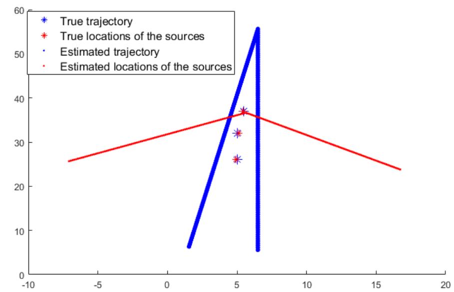

Fig. 5: Linearly transformed source locations and trajectory to best fit the true source locations and trajectory respectively, when the samples are corrupted by additive white Gaussian noise of SNR=20dB.

A. ESTIMATING THE CENTERS OF GAUSSIAN FUNCTIONS FROM THEIR SUM

The measurements we obtain from K sources are:

$$f_j(nT) := f_{j,n} = \sum_{k=1}^{K} A_{k,j} e^{-(l_j nT - Y_{k,j})^2}. \tag{13}$$

Leveraging the results in [11, 12], we can find coefficients $c_{m,n}$ that allow us to approximately reproduce exponentials, as follows:

$$\sum_{n\in\mathbb{N}} c_{m,n} e^{-(t-nT)^2} \approx e^{j\omega_m t}, \tag{14}$$

for $\omega_m = \omega_0(1 - \frac{2}{2K-1}m)$, $m = 0, 1, ..., 2K-1$, where K is the number of sources we aim to estimate and ω_0 is arbitrary.

Then, we can multiply the measurements $f_{j,n}$ by the coefficients $c_{m,n}$, to obtain the *signal moments*:

$$\begin{aligned} s_m &= \sum_n c_{m,n} f_{j,n} = \sum_n c_{m,n} \sum_{k=1}^{K} A_{k,j} e^{-(l_j nT - Y_{k,j})^2} \\ &= \sum_{k=1}^{K} A_{k,j} \sum_n c_{m,n} e^{-(l_j nT - Y_{k,j})^2} \\ &\overset{(a)}{\approx} \sum_{k=1}^{K} A_{k,j} e^{j\omega_m Y_{k,j}}, \end{aligned} \tag{15}$$

where (a) follows from Eq. (14).

Finally, we use Prony's method [13] on s_m, to obtain the frequency components $Y_{k,j}$, as well as the terms $A_{k,j}$, for each line j and source $\mathbf{S}_k$.

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