On the Multidimensional Extension of the Quincunx Subsampling Matrix

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***Abstract*—The dilation matrix associated with the three-dimensional (3-D) face-centered cubic (FCC) sublattice is often considered to be the natural 3-D extension of the two-dimensional (2-D) quincunx dilation matrix. However, we demonstrate that both dilation matrices are of different nature: while the 2-D quincunx matrix is a similarity transform, the 3-D FCC matrix is not. More generally, we show that is impossible to obtain a dilation matrix that is a similarity transform and performs downsampling of the Cartesian lattice by a factor of two in more than two dimensions. Furthermore, we observe that the popular 3-D FCC subsampling scheme alternates between three different lattices: Cartesian, FCC, and quincunx. The latter one provides a less isotropic sampling density, a property that should be taken into account to properly orient 3-D data before processing using such a subsampling matrix.**

***Index Terms*—Nonseparable design, two-channel filterbanks, wavelet decomposition, 2-D quincunx sampling, 3-D FCC sampling.**

I. Introduction

THE QUINCUNX sampling scheme is an interesting configuration that is commonly used for designing critically sampled filterbanks and discrete wavelet transforms in two dimensions [1]–[7]. In contrast to the dyadic separable case, the quincunx dilation matrix leads to a two-channel filterbank architecture—or equivalently one single wavelet—and reduces the scale more progressively: a factor $\sqrt{2}$ instead of 2. We briefly review the two-dimensional (2-D) quincunx scheme in Section II.

The growing availability of three-dimensional (3-D) volumetric and spatio-temporal data increases the interest for 3-D two-channel designs. The dilation matrix associated to the 3-D face-centered cubic (FCC) sublattice is often proposed as the natural extension of the 2-D quincunx case [7]–[9]. However, we point out a fundamental difference between the 2-D quincunx and the 3-D FCC matrices: the 3-D FCC matrix does not correspond to a similarity transform (i.e., an angle-preserving transformation such as rotation or symmetry combined with dilation) as does the 2-D quincunx matrix. Moreover, we show that, for more than two dimensions, it impossible to find a dilation matrix on the Cartesian lattice that is a similarity transform and at the same time leads to a two-channel filterbank. Consequently, when the 3-D FCC dilation matrix is cascaded, the Voronoi cells of the sublattices are not just obtained by a simple rotation/symmetry and dilation. In particular, for the 3-D FCC case, we observe an alternation between three types of lattices: Cartesian, FCC, and quincunx in the $x-z$ plane. This observation can be useful to guide the orientation of the 3-D data before processing. These results are discussed in Section III.

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II. Two-Dimensional Quincunx Subsampling

We characterize the Cartesian lattice by its sites $\mathbf{k}=[k_1\ k_2]^{\mathrm{T}}$, $k_1, k_2 \in \mathbb{Z}$. An admissible 2×2 dilation matrix $\mathbf{D}$, mapping $\mathbf{k}$ to $\mathbf{Dk}$, must satisfy the following properties [10], [11].

1) The new lattice forms a sublattice of the Cartesian lattice; a trivial property when each element of $\mathbf{D}$ is integer.
2) The magnitude of each eigenvalue λ_i of $\mathbf{D}$ must be strictly larger than 1 to ensure a dilation in each dimension.

After applying the dilation matrix, the density of the lattices sites is reduced by a factor $|\det \mathbf{D}|$. The Voronoi cell of the lattice is defined as the set of points closer to the origin than to any other lattice site, and completely characterizes the lattice's geometric structure [12]. The dual or reciprocal lattice is characterized by the matrix that is the transpose of the inverse dilation matrix; the reciprocal lattice serves in the frequency domain to replicate the spectrum while its Voronoi cell can be considered as the Nyquist region [8], [13].

The quincunx sublattice consists of those lattices sites $[k_1\ \ k_2]^{\mathrm{T}}$, where $k_1+k_2=2n$, n integer. There are two common choices for the corresponding dilation matrix:

$$\mathbf{D}_1=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix},\quad \mathbf{D}_2=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}. \tag{1}$$

Note that both matrices correspond to a similarity transform; i.e., a symmetry ($\mathbf{D}_1$) or a rotation ($\mathbf{D}_2$), combined with a dilation that reduces the number of sites by a factor of two ($|\det \mathbf{D}_1|=|\det \mathbf{D}_2|=2$). For signal processing applications, an interesting property is to have the iterated sublattice coincide again with a dilated Cartesian lattice after a few iterations. This property also contributes to the construction of smooth wavelets [14]. For these examples, we have $\mathbf{D}_1^2=2\mathbf{I}$ and $\mathbf{D}_2^8=16\mathbf{I}$.

The design of orthogonal filters for quincunx sampling is strongly determined by the shape of the Voronoi cell of the reciprocal sampling lattice. More specifically, the goal is to design an orthogonal filter whose frequency response is reasonably close to the ideal filter characterized by the indicator function of the Nyquist region of the sampling lattice in the frequency domain, which correspond to the Voronoi cell of the reciprocal lattice. Some filter and wavelet design examples can be found in [5], [7], [10], and [15]–[19].

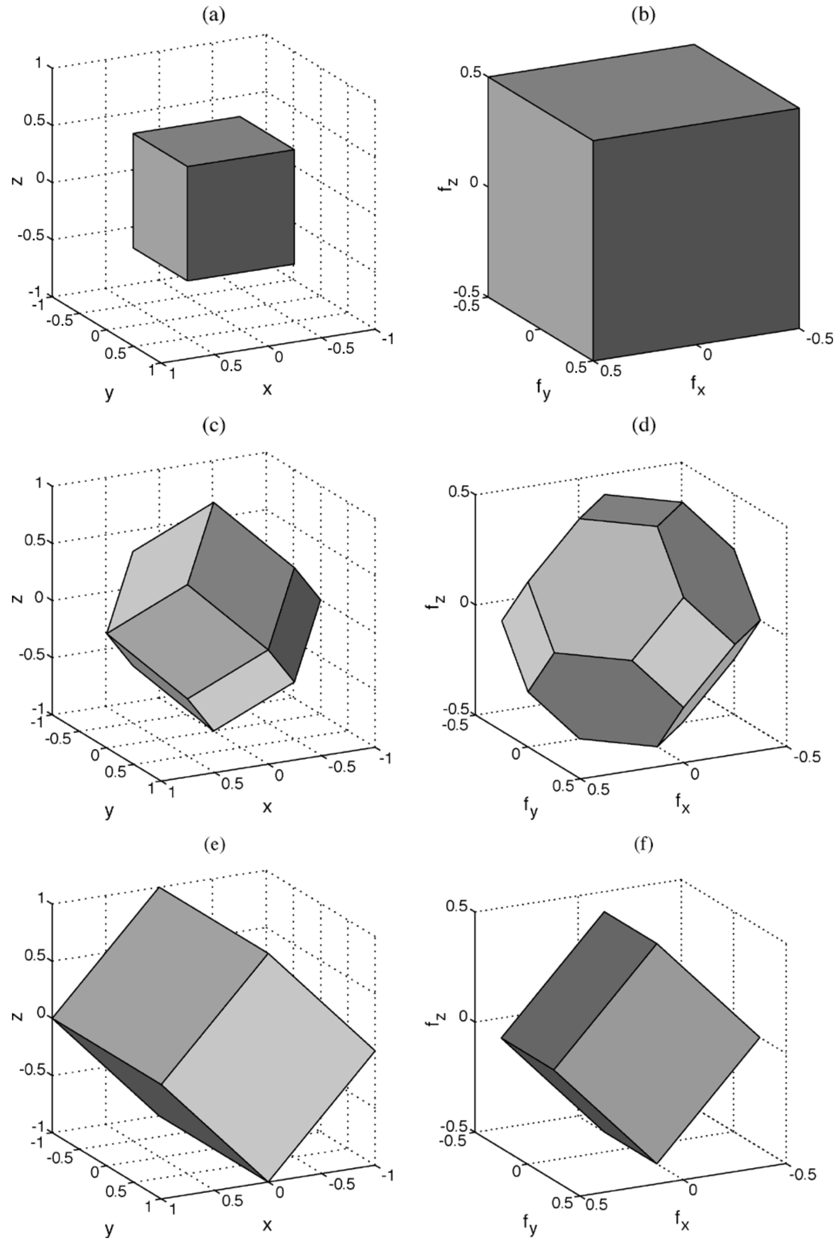

Fig. 1. Types of 3-D lattices encountered when the 3-D FCC dilation matrix is iterated. Left column: the Voronoi cell of the lattice in the spatial domain for the lattices (a) $\mathbf{D}^0\mathbf{k}$, (c) $\mathbf{D}^1\mathbf{k}$, (e) $\mathbf{D}^2\mathbf{k}$ (notice that the side in the $x-z$ plane have length $\sqrt{2}$, while the sides in the y direction have length 2). Right column: the Voronoi cell of the reciprocal lattice in the frequency domain for the lattices (b) $(\mathbf{D}^{-\mathrm{T}})^0\mathbf{k}$, (d) $(\mathbf{D}^{-\mathrm{T}})^1\mathbf{k}$, (f) $(\mathbf{D}^{-\mathrm{T}})^2\mathbf{k}$. (a) Cartesian lattice; (b) cartesian lattice; (c) FCC lattice; (d) BCC lattice; (e) quincunx lattice $x-z$; (f) quincunx lattice $f_x - f_z$.

Since the 2-D quincunx matrix corresponds to a similarity transform, its associated Voronoi cell in the spatial and the frequency domain is simply rotated or mirrored and dilated. Therefore, iterating $\mathbf{D}_1$ and $\mathbf{D}_2$ alternates between a Cartesian and a quincunx sublattice.

III. Three-Dimensional FCC Subsampling

The dilation matrix corresponding to the 3-D FCC sublattice is often considered to be the natural extension of the 2-D quincunx scheme. The corresponding most popular dilation matrix (cf. [4], [8], [10], and [20]) is

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix}. \qquad (2)$$

It leads to a two-channel design $(|\det \mathbf{D}| = 2)$ and coincides with a dilated Cartesian lattice after three iterations; i.e., we have $\mathbf{D}^3 = 2\mathbf{I}$.

The use of the FCC sampling scheme is motivated by the energy packing argument [21]. Indeed, together with the body-centered cubic (BCC) lattice, the FCC lattice has the densest lattice packing in 3-D: it retains a maximal energy proportion for signals with a spherical spectrum.[1] Therefore, as in the 2-D case, the shape of the frequency response of an orthogonally designed filter on the FCC lattice tends to the indicator function

[1]In signal processing literature, the term face-centered orthorhombic (FCO) is often used instead of FCC. However, the FCO lattice allows for a scaling along the coordinate axes and therefore the argument of maximal energy compaction for signals with a spherical spectrum is no longer valid. Since the scaling is rarely used, we prefer the use of the term FCC.

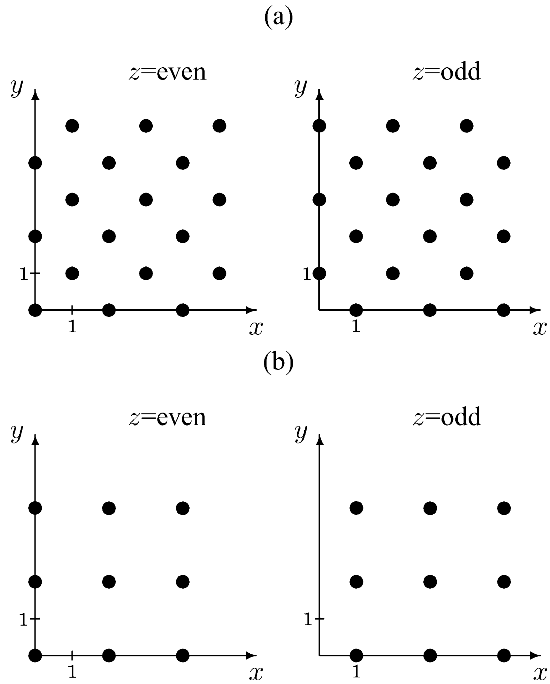

Fig. 2. Two-dimensional cuts of the 3-D lattices obtained by iterating the 3-D FCC dilation. (a) $\mathbf{D}\mathbf{k}$. (b) $\mathbf{D}^2\mathbf{k}$.

TABLE I
PROPERTIES OF THE ITERATED 3-D FCC MATRIX OF (2)

iteration	spatial lattice	reciprocal lattice	sampling density
$3n$	Cartesian lattice	Cartesian lattice	$2^{3n}(1,1,1)$
$3n+1$	FCC lattice	BCC lattice	$2^{3n}(\sqrt[3]{2}, \sqrt[3]{2}, \sqrt[3]{2})$
$3n+2$	Quincunx (x-z)	Quincunx (f_x-f_z)	$2^{3n}(\sqrt{2}, 2, \sqrt{2})$

of the Voronoi cell of the BCC lattice, which is a space-filling polyhedron that fills the space in an optimal way for signals with a spherical spectrum. Some examples of 3-D filter design can be found in [10], [17], [19], [20], and [22]–[24].

However, there is a fundamental difference between the 2-D quincunx and the 3-D FCC dilation matrix. The latter does not correspond to a similarity transform. More generally, we can show the following.

Proposition 1: For dimensions greater than 2, no dilation matrix can be a similarity matrix with determinant 2.

Proof: Assume that $\mathbf{D}$ is an admissable dilation matrix in N dimensions that corresponds to a similarity transform. Then it should satisfy

$$\mathbf{D}^{\mathrm{T}}\mathbf{D} = m\mathbf{I}, \tag{3}$$

where m is an integer. Taking the determinant of (3) gives

$$|\det \mathbf{D}| = \sqrt{m^N}. \tag{4}$$

On the other hand, by the two-channel design constraint, we need

$$|\det \mathbf{D}| = 2. \tag{5}$$

Satisfying both constraints (4) and (5) would require

$$m = \sqrt[N]{4} \tag{6}$$

with m integer, which is only possible for $N = 2$. ■

Now we know that no 3-D dilation matrix with determinant 2 exists that corresponds to a similarity transform. Nevertheless, it is interesting to take a closer look at the popular 3-D FCC matrix (2) and the consequences of not having this property. For this purpose, we determine the Voronoi cell at each iteration for both the spatial and the dual lattice [12], [21]. Since we have $\mathbf{D}^3 = 2\mathbf{I}$, the iterated scheme alternates between three lattices, but due to the lack of being a similarity transform, with fundamentally different Voronoi cells. For $\mathbf{D}^{3n}$, $n \in \mathbb{N}$, we have the Cartesian lattice and the corresponding cubic Voronoi cell in the spatial and the frequency domain, see Fig. 1(a) and (b). For $\mathbf{D}^{3n+1}$, we have the FCC lattice in the spatial domain [Fig. 1(c)] where the Voronoi cell is a rhombic dodecahedron, and the BCC lattice in the frequency domain [Fig. 1(d)] where the Voronoi cell is a truncated octahedron. Finally, for $\mathbf{D}^{3n+2}$, we obtain a quincunx sublattice in the $x - z$ plane, while the y dimension is already subsampled by 2; see Fig. 1(e) and (f), respectively. Logically, the Voronoi cell of the spatial domain gets larger when the lattice gets coarser, while the inverse happens for the dual lattice. The lattices are also illustrated by 2-D cuts in Fig. 2. The alternation between those lattices is summarized in Table I and has two important consequences. First, the optimal energy packing argument for orthogonal filter design is only valid for one of the three lattices. Second, the quincunx arrangement for each third lattice $\mathbf{D}^{3n+2}$ has a less isotropic sampling density, which also influences orthogonally designed filters. Applications can use this property to properly adapt the orientation of 3-D data before treatment; e.g., for spatio-temporal data, the temporal direction could be put along the y-axis.

IV. CONCLUSION

The 3-D FCC dilation matrix is often considered to be the natural extension of the 2-D quincunx case. However, the dilation matrices have a different nature. In this paper, we have shown that the popular 3-D FCC dilation matrix, as opposed to the 2-D quincunx matrix, does not correspond to a similarity transform. Moreover, for more than two dimensions, it is impossible to build a two-channel dilation matrix on the Cartesian lattice which corresponds to a similarity transform. Still, the 3-D FCC dilation matrix can be useful to treat 3-D volumetric or spatiotemporal data. However, one should be aware that iterating the dilation matrix produces an alternation between lattices with a different degree of isotropy; i.e., Cartesian, FCC, and quincunx lattices.

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