Iterated Rational Filter Banks – Underlying Limit Functions

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The term 'Rational Filter Bank' (RFB) stands for 'Filter Bank with Rational Rate Changes'. An analysis two-band RFB critically sampled is shown with its synthesis counterpart in figure 1. G stands typically for a low-pass FIR filter, whereas H is high-pass FIR. We are interested, in this paper in the iteration of the sole low-pass branch, which leads, in the integer case ($q = 1$), to a wavelet decomposition.

Kovačević and Vetterli [1] have wondered whether iterated RFB could involve too, a discrete wavelet transform. Actually, Daubechies proved that whenever p/q is not an integer and G is FIR, this could not be the case. We here show that despite this discouraging feature, there still exists, not only one function (then shifted), as in the integer case, but an infinite set of compactly supported functions $\varphi_s(t)$. More importantly, under certain conditions, these functions appear to be 'almost' the shifted version of one sole function. These φ_s are constructed the same way as in the dyadic case ($p = 2$, $q = 1$), that is to say by the iteration of the low-pass branch of a synthesis RFB, but in this case the initialization is meaningfull. More precisely, let $X_j(z)$ be the result at iteration j of the process, where $X_0(z) = z^s$, then one finds

$$x_j[n] = g_j[nq^j - sp^j]$$

with

$$G_j(z) = G(z^{p^{j-1}})G(z^{qp^{j-2}})G(z^{q^2p^{j-3}})\ldots G(z^{q^{j-1}})$$

Under certain conditions, the set of functions $x_j[\mathrm{Int}(t(p/q)^j)]$ (wher $\mathrm{Int}(x)$ denotes the integer part of x) converges when the number of iterations j tends to infinity, toward the limit function $\varphi_s(t)$. As warned supra, we do not have $\varphi_s(t) = \varphi_0(t - s)$ as we used to in the integer case.

We now list some properties obeyed by these functions: most of them are very similar to the ones we know from the integer case.

- the support of φ_s is included in $[s + l/(p - q), s + L/(p - q)]$ if $G(z) = \sum_{k=l}^{L} g[k]z^k$
- For convergence to be achieved, it is necessary that $G(1) = p$ and $G(e^{2i\pi k/p}) = 0$ for $k = 1 \ldots p - 1$.
- φ_s obey a *two-scale equation* (for $q = 1$ see [2])

$$\varphi_s(t) = \sum_k g[kq - sp]\varphi_k\left(\frac{p}{q}t\right)$$

  which reduces to the classical two-scale difference equation, when one assumes that $\varphi_s(t) = \varphi_0(t - s)$.

- Assume $G(z) = q/p(1 - z^p)/(1 - z^q)F(z)$, then if φ_s is generated by G, and $\varphi^!_s$ by F, one has the relation

$$\varphi'_s(t) = -\varphi^!_{s+1}(t) + \varphi^!_s(t)$$

  This in particular indicates that the multiplicity of factors $(1 - z^p)/(1 - z)$ in G generates regularity for the associated functions.

- Back to the analysis scheme of figure 1. We name Y_j the result, after $j - 1$ iterations of the low-pass filter and one iteration of the high-pass, φ_s the function associated to $G(1/z)$ and $\psi_s(t) = \sum_k h[np - kq]\varphi_k(p/qt)$. Then

$$y_j[n] = \int x(t)\psi_n(t(p/q)^{-j})\,dt$$

  where $x(t)$ is any function such that $X_0[n] = \int x(t)\varphi_n(t)\,dt$. This is not in general a discrete wavelet transform since $\psi_n(t) \neq \psi_0(t - n)$ (in the dyadic case see [3]).

- The reconstruction scheme is associated to a mirror set of limit functions $\tilde{\varphi}_s$ and $\tilde{\psi}_s$ which obey $\int \tilde{\varphi}_n\varphi_{n'} = \int \tilde{\psi}_n\psi_{n'} = \delta[n - n']$ and $\int \tilde{\varphi}_n\psi_{n'} = \int \tilde{\psi}_n\varphi_{n'} = 0$.

The important point to note is that if the set of functions is regular enough (enough many derivates are continuous) then it appears that the set 'almost' recovers the shift property $\varphi_s(t) \simeq \varphi_0(t - s)$ (see figure 2). This point has not yet been totally made clear, but is indeed a very interesting feature in order to, for instance, use it as a basis for a fast wavelet transform computation (in the dyadic case, see [4,5]). More generally the iterated RFB are fast constant Q time-frequency transforms that allow finer scale resolution (the scale parameter is here p/q which can be made as close to 1 as one wants).

References

[1] J.Kovačević et M. Vetterli *Perfect Reconstruction Filter Banks with Rational Sampling Rate Changes* ICASSP 1991

[2] I. Daubechies et J. Lagarias *Two-Scale Difference Equations 1. Existence and Global Regularity of Solutions* SIAM J. MATH. ANAL., Vol. 22,No. 5, pp. 1388–1410, Septembre 1991

[3] S. Mallat *A Theory for Multiresolution Signal Decomposition: The Wavelet Decomposition* IEEE TRANS. PATTERN ANAL. MACHINE INTELL., Vol. 11,No. 7, Juillet 1989

[4] M. Holschneider, R. Kronland-Martinet, L. Morlet et Ph. Tchamitchian *A Real-Time Algorithm for Signal Analysis with the Help of the Wavelet Transform* WAVELETS,TIME-FREQUENCY METHODS AND PHASE SPACE, J.M. Combes A.Grossman Ph Tchamitchian eds, Spriger-Verlag

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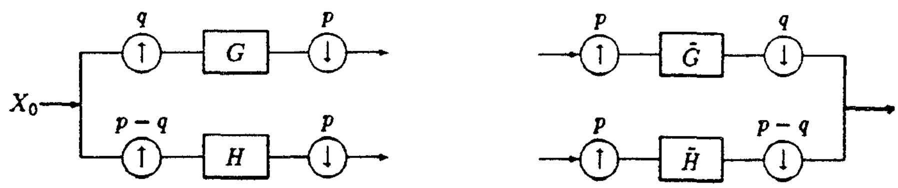

Figure 1: Analysis and synthesis two-band rational filter banks

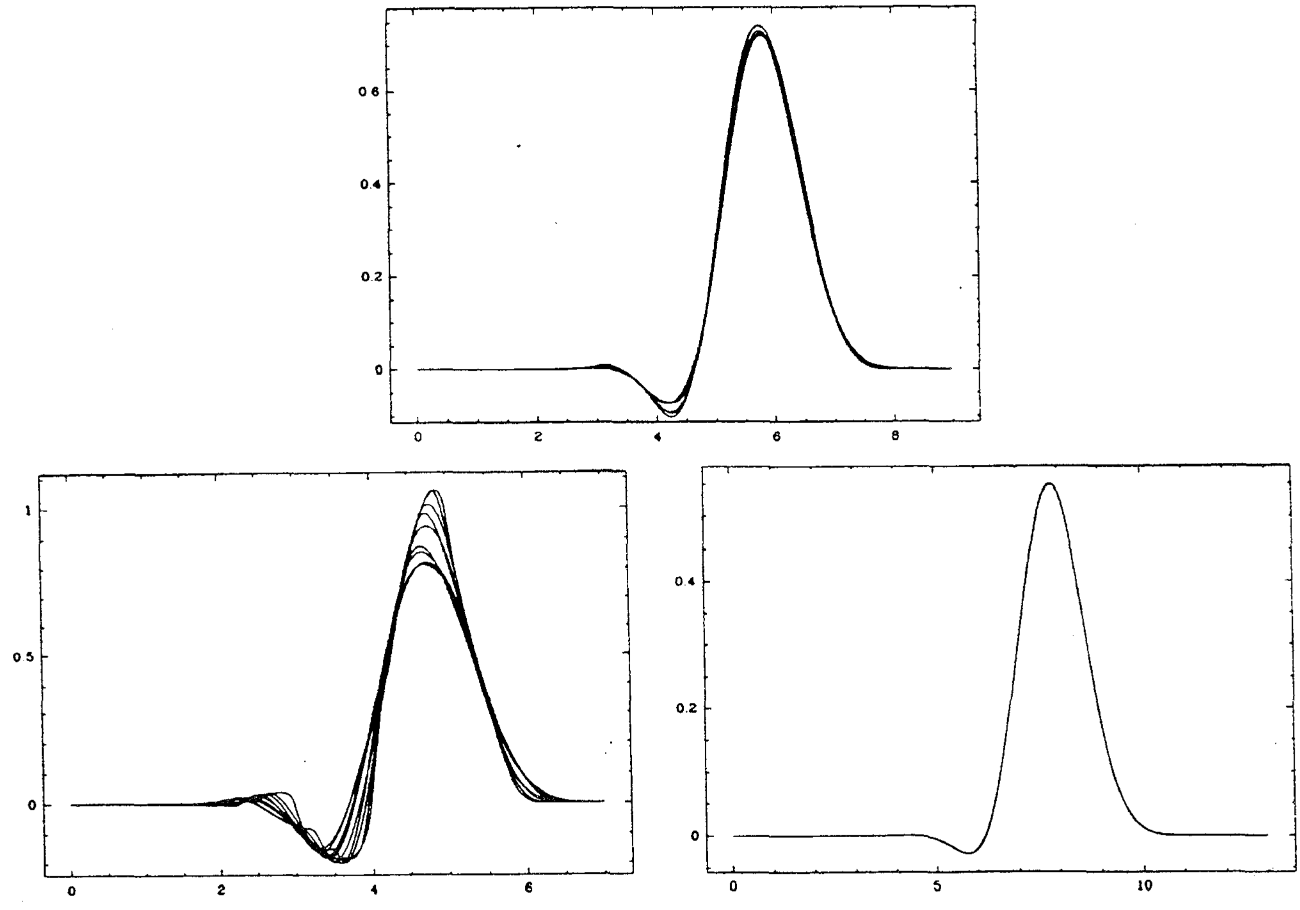

Figure 2: $\varphi_n(x-n)$ for $n = 0 \ldots 7$ generated by (case p,q=3,2) **left** $G(z) = \frac{1}{9}\left(\frac{z^3-1}{z-1}\right)^3 (2z-1)$

top $G(z) = \frac{1}{9}\left(\frac{z^3-1}{z-1}\right)^4 (2z-1)$

right $G(z) = \frac{1}{243}\left(\frac{z^3-1}{z-1}\right)^6 (2z-1)$