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Comparison of wavelets from the point of view of their approximation error

Michael Unser and Thierry Blu

Biomedical Imaging Group,
Swiss Federal Institute of Technology (EPFL)
DMT/IOA, P.O. Box 127
CH-1015 Lausanne, Switzerland

Email: {Michael.Unser,Thierry.Blu}@epfl.ch

ABSTRACT

We present new quantitative results for the characterization of the L_2-error of wavelet-like expansions as a function of the scale a. This yields an extension as well as a simplification of the asymptotic error formulas that have been published previously. We use our bound determinations to compare the approximation power of various families of wavelet transforms. We present explicit formulas for the leading asymptotic constant for both splines and Daubechies wavelets. For a specified approximation error, this allows us to predict the sampling rate reduction that can obtained by using splines instead Daubechies wavelets. In particular, we prove that the gain in sampling density (splines vs. Daubechies) converges to π as the order goes to infinity.

Keywords: wavelet transform, linear approximation, error analysis, approximation theory, asymptotics, splines, Daubechies wavelets.

1. INTRODUCTION

The order of approximation is a fundamental notion in wavelet theory. It is a concept that is intimately related to the way in which wavelets interact with polynomials. It has important implications for data compression [1] (i.e., the ability of wavelets to approximate piecewise smooth functions), and signal analysis [2, 3] (i.e., the characterization of local singularities from the decay of the wavelet coefficients across scale).

There are four equivalent indicators of the order of the transform:

(*i*) The largest L such that the synthesis refinement filter can be factorized as

$$H(z) = (1+z^{-1})^L Q(z) \tag{1}$$

where $Q(z)$ is bounded on the unit circle.

(*ii*) The number of vanishing moments of the analysis wavelet $\tilde{\psi}(x)$; i.e.,

$$\int_{-\infty}^{+\infty} x^n \tilde{\psi}(x)dx = 0, \quad (n = 0,\ldots,L-1) \tag{2}$$

(*iii*) The order of the synthesis scaling function $\varphi(x)$ in the sense of the reproduction of polynomials of degree n=L-1. In other words, there must exist sequences $c_n(k)$ such that

$$\sum_{k\in Z} c_n(k)\varphi(x-k) = x^n, \quad (n = 0,\ldots,L-1). \tag{3}$$

(*iv*) The rate of decay of the approximation error $\left\| f - \tilde{P}_{2^i} f \right\|$ as the scale $a = 2^i$ goes to zero:

$$\left\|f-\tilde{P}_a f\right\| = O(a^L), \text{ as } a=2^i \to 0 \text{ (or } i \to -\infty), \quad (4)$$

where $\tilde{P}_{2^i} f$ denotes a (biorthogonal) projection of f into the multiresolution subspace $V_i = \mathrm{span}\{\varphi_{i,k}\}_{k\in Z}$.
The first two conditions are emphasized in most wavelet textbooks [4, 5, 6, 7]. The equivalence with conditions (*iii*) and (*iv*) is not quite as well known; it goes back to the Strang-Fix theory of approximation [8].

In this paper, we are especially interested in the comparison of wavelets from the point of view of their approximation power, that is, the way in which the linear approximation error $\left\|f-\tilde{P}_{2^i} f\right\|$ decays as a function of the scale. While all Lth order wavelets yield the $O(a^L)$ decay (4), we want to be more quantitative and look at the magnitude of the error more closely.

This kind of investigation was initiated by Sweldens who derived some upper bound constant for the asymptotic error which he used to compare the various wavelet transforms available [9, 10]. This error analysis was then refined by Unser who derived the exact asymptotic form of the error [11]. These analyses were all carried out in the time-domain using the Taylor series as the main tool. While this kind of approach yields a local (pointwise) characterization of the error, it does not lend itself easily to the derivation of global error estimates, except in the asymptotic regime. In this paper, we will circumvent this difficulty by using some recent theorems in approximation theory [12]. In particular, we will show how it is possible to perform the entire error analysis in the Fourier domain. This will allow us to obtain simpler and more reliable error estimates. In particular, we will use our bound determinations to quantify the approximation power of various wavelet transforms.

2. PRELIMINARY NOTIONS

We take a more general perspective than the usual multiresolution formulation and consider signal representations in terms of rescaled translates of an (almost) arbitrary generating function φ. At this stage, φ is not required to have the multiresolution property and the scale parameter (or sampling step) a can be arbitrary (not necessarily a power of two).

Associated with the function φ is the autocorrelation sequence $a_\varphi(k) = \langle \varphi(x), \varphi(x-k)\rangle$ whose z-transform is

$$A_\varphi(z) = \sum\nolimits_{k\in Z} \langle \varphi(x), \varphi(x-k)\rangle z^{-k}. \quad (5)$$

Its Fourier transform may be determined either by replacing z by $e^{j\omega}$, or by using the identity

$$A_\varphi(e^{j\omega}) = \sum_{k\in Z} \left|\hat{\varphi}(\omega+2\pi k)\right|^2, \quad (6)$$

which is a direct consequence of Poisson's summation formula; we use the mathematical symbolism $\hat{f}(\omega)$ to denote the Fourier transform of $f(x)$.

Definition 2.1 : $\varphi(x)$ is an Lth order *generating* function if and only if

(a) $0 < A \le A_\varphi(e^{j\omega}) \le B < +\infty$ (Riesz basis condition)

(b) $\hat{\varphi}(0) = 1$ and $\hat{\varphi}^{(m)}(2\pi k) = 0,\ k\in Z,\ k\ne 0$ for $(m = 0,\ldots,L-1)$ (Order property)

where $\hat{\varphi}(\omega)$ is the Fourier transform of φ, and $\hat{\varphi}^{(m)}(\omega)$ denotes its mth derivative with respect to ω.
The corresponding signal representation space at scale a is

$$V_a(\varphi) = \left\{ f_a(x) = a^{-1/2} \sum_{k\in Z} c_a(k)\varphi(x/a-k) : c_a \in l_2 \right\}. \quad (7)$$

Condition (a) ensures that $V_a(\varphi)$ is a well-defined subspace of L_2 and that each function $f_a \in V_a(\varphi)$ has a unique representation in terms of its coefficients $c_a(k)$ [13]. The constraint (b) is the Fourier domain equivalent of Condition (iii) given in the introduction; it ensures that φ reproduces all polynomials of degree L-1. This equivalence is well known in approximation theory; it is often referred to as the Strang-Fix conditions [14, 15].

To define a linear approximation operator in $V_a(\varphi)$, we also need to specify an *analysis* function $\tilde{\varphi}$. We assume that this function (which is not necessarily in L_2) has a bounded Fourier transform and is biorthogonal to φ in the sense that

$$\langle \varphi(x-k), \tilde{\varphi}(x-l) \rangle = \delta_{k-l} \tag{8}$$

The corresponding approximation operator is

$$\left(\tilde{P}_a f\right)(x) = a^{-1} \sum_{k \in Z} \langle f, \tilde{\varphi}(x/a-k) \rangle \cdot \varphi(x/a-k). \tag{9}$$

It can be shown that $\tilde{P}_a f$ is a projector. Note that it is an orthogonal projector if and only if $\tilde{\varphi} \in V(\varphi)$, in which case $\tilde{\varphi} = \varphi_d$ is the so-called dual of φ, which is unique [13].

In the case of a wavelet transform, there is an additional multiresolution constraint: the analysis and synthesis functions $\tilde{\varphi}$ and φ must be solution of two-scale equations.

Definition 2.2 : $\varphi(x)$ is an Lth order *scaling function* if and only if it is an Lth order generating function satisfying the additional two-scale relation :

$$\varphi(x/2) = \sum_{k \in Z} h(k) \varphi(x-k). \tag{9}$$

By applying this refinement equation *ad infinitum*, one gets the equivalent infinite product representation of the Fourier transform of the scaling function φ

$$\hat{\varphi}(\omega) = \prod_{i=1}^{+\infty} \left(\frac{H(e^{j\omega/2^i})}{2} \right), \tag{10}$$

where $H(z)$ denotes the z-transform of the refinement filter h. It is then possible to establish the following which gives the equivalence between the conditions (i) and (iii) mentioned in the introduction (cf. [1]).

Proposition 2.3: $H(e^{j\omega})$ has zeros of multiplicity N at $\omega = \pi$ (i.e., $H(z) = 2^{-L}(1+z)^L \cdot Q(z)$ where $Q(z)$ is a stable transfer function) if and only if φ is an Lth order scaling function.

3. THEORETICAL RESULTS

Here, we are interested in studying the behavior of the L_2-approximation error

$$\|f - P_a f\| = \left(\int_{-\infty}^{+\infty} (f(x) - P_a f(x))^2 dx \right)^{1/2}, \tag{11}$$

as a function of the scale a. In particular, if we consider the dyadic scale $a = 2^i$, it is clear that this measure also represents the truncation error for a wavelet approximation in which all the finer coefficients have been set to zero. We will derive estimates for this error in order to compare the performance of the various wavelet transforms available and to predict their coding performance.

3.1 Approximation theorem

Most of our error estimates involve a Fourier kernel associated with the projector; it is defined as follows:

$$E(\omega) = 1 - \frac{|\hat{\varphi}(\omega)|^2}{A_\varphi(\omega)} + A_\varphi(\omega) \left| \hat{\tilde{\varphi}}(\omega) - \frac{\hat{\varphi}(\omega)}{A_\varphi(\omega)} \right|^2, \tag{12}$$

where $\hat{\varphi}(\omega)$ and $\hat{\tilde{\varphi}}(\omega)$ are the Fourier transforms of φ and $\tilde{\varphi}$, respectively — $A_\varphi(e^{j\omega})$ is given by (6). Note that the right most term in (12) is zero in the case of a semi-orthogonal transform, that is, when $\tilde{\varphi} \in V(\varphi)$. Our main result is as follows (cf. [2]):

Theorem 3.1: *For all $f \in W_2^r$ (Sobolev's space of order $r > \frac{1}{2}$), the approximation error is given by*

$$\|f - \tilde{P}_a f\| = \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} E(a\omega)\left|\hat{f}(\omega)\right|^2 d\omega\right]^{1/2} + \underbrace{\gamma a^r \left\|f^{(r)}\right\|_{L_2}}_{\Delta e_f}. \tag{13}$$

where $|\gamma| \le \frac{2}{\pi^r}\sqrt{\zeta(2r)\|E\|_\infty}$ with $\zeta(s) = \sum_{n\ge 1}\frac{1}{n^s}$ (Riemann Zeta function).

Thus, the approximation error is made up of two terms: a first one that is easy to compute by integration of the spectrum of f against the kernel $E(a\omega)$, and a second one, Δe_f, that is bounded and depends on the regularity (differentiability) of f. Clearly, the second term will vanish whenever the function f to approximate is sufficiently smooth with respect to the sampling step, that is, when r is large and a small (oversampling). Another remarkable property is that the second term — which may be positive or negative — will cancel out when one computes the average error over all possible shifts of the input function f (cf. Theorems 2 and 3 [16]). Hence, the first part of the error in (13) provides a valid distortion measure which is meaningful whenever the initial positioning of f with respect to the sampling grid can be considered to be arbitrary — this turns out to be the case in most applications. In addition, $\Delta e_f(a)$ also vanishes when the sampling does not introduce any aliasing; for instance, when either f or ($\hat{\tilde{\varphi}}$ and $\hat{\varphi}$) are bandlimited.

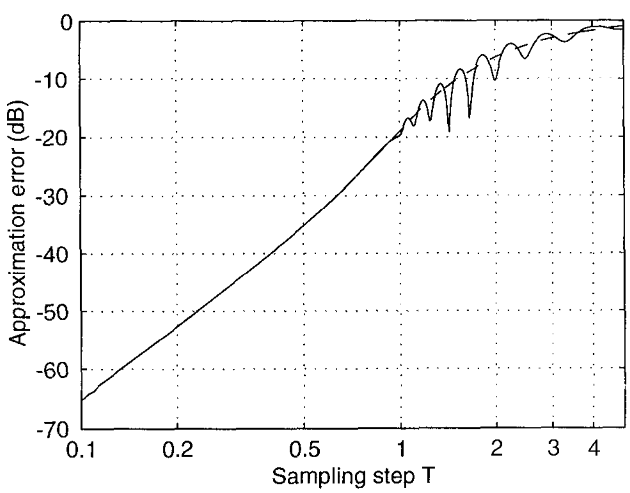

Fig.1. Least squares approximation error of the function $f(x) = e^{-x^2/2}$ by cubic splines as a function of the sampling step T. The error estimate $\varepsilon_f^2(T)$ (dashed line) is an unbiased smoothed version of the true error (solid line).

3.2 Practical error estimation

From the above theorem, we have the following measure of the approximation error

$$\varepsilon_f^2(a) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} E(a\omega)\left|\hat{f}(\omega)\right|^2 d\omega \cong \left\|f - \tilde{P}_a f\right\|^2. \tag{14}$$

This is a very good estimate of the true approximation error for the reasons that have just been mentioned. It is also very easy to compute if we know the spectrum of $f(x)$. The graph in Fig. 1 provides an example of comparison between our error estimate (left hand side of (14)) and the true approximation error (right hand side of (14)). In this experiment, we computed the least squares approximation of the test function $f(x) = e^{-x^2/2}$ using cubic splines with step size T. The two curves are in very close agreement with each other which demonstrates the adequacy of the error estimate (14). Also note that the measure $\varepsilon_f^2(a)$ yields a smoothed version of the true approximation error (averaging property).

Thus, the kernel $E(\omega)$ contains all the necessary information for assessing the performance of a given approximation operator. Graphing these kernels will give us a direct and easy way of comparing the performance of various wavelet transforms. Some examples of least squares spline error kernels are shown in Fig. 2.

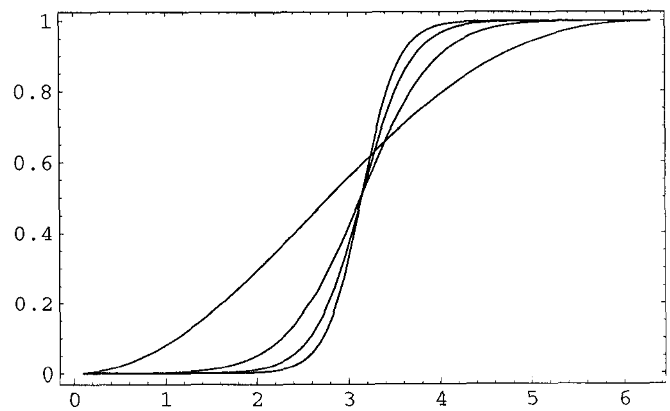

Fig.2. Frequency plot of the error kernels for the orthogonal projection on the spline spaces of degree 0, 1, 2, and 3. The higher order kernels are consistently smaller below the Nyquist frequency $\omega = \pi$. This explains why higher order splines provide a better approximation of functions that are predominantly lowpass.

3.3 Asymptotic error analysis

We will now use Theorem 1 to compute the asymptotic form of the approximation error as the scale gets sufficiently small. This will allow us to add higher order terms to the first order result in [11]. An advantage of the present formulation is that it is much more direct than the time-domain derivation that was initially proposed in [11].

For f smooth (or a sufficiently small), we have $a^r\|f^{(r)}\| \to 0$ which means that the second term in Theorem 1 vanishes. Therefore

$$\|f - \tilde{P}_a f\|^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} E(a\omega)|\hat{f}(\omega)|^2 d\omega, \quad \text{as } a^r\|f^{(r)}\| \to 0 \tag{15}$$

To study the behavior of this error as $a \to 0$, we take the Taylor series of $E(\omega)$ at $\omega = 0$:

$$E(\omega) = \sum_{k=0}^{r_0} \frac{E^{(2k)}(0)}{(2k)!} a^{2k}\omega^{2k} + O(a^{2r_0+2}\omega^{2r_0+2})$$

Substitution in (15) yields

$$\|f - \tilde{P}_a f\|^2 = \sum_{k=0}^{r_0} C_k a^{2k}\|f^{(k)}\|^2 + O(a^{2r_0+2}) \tag{16}$$

where

$$\|f^{(k)}\|^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \omega^{2k}|\hat{f}(\omega)|^2 d\omega \tag{17}$$

is the energy of the kth derivative of f, and where

$$C_k = \frac{E^{(2k)}(0)}{(2k)!}. \tag{18}$$

Let us now consider an Lth order transform which preserves all polynomials up to order n=L-1. One implication of this property (in addition to the vanishing moments of the analysis wavelet) is that $E(\omega)$ must be essentially vanishing at the origin in the sense that $E(\omega) = O(\omega^{2L})$, as $\omega \to 0$. Thus, by keeping the first non-vanishing term in (16) only, we obtain

$$\|f - \tilde{P}_a f\| = C_\varphi^- a^L \|f^{(L)}\|, \text{ as } a \to 0 \tag{18}$$

where $C_\varphi^- = \sqrt{E^{(2k)}(0)/(2L)!}$; this is precisely the result reported in [11]. By differentiating the kernel $E(\omega)$ and using the biorthogonality property, it is then not difficult to obtain the explicit form of the leading asymptotic constant

$$C_\varphi^- = \frac{1}{L!}\left(\sum_{\substack{k\in Z\\ k\neq 0}} |\hat{\varphi}^{(L)}(2\pi k)|^2\right)^{1/2}, \tag{19}$$

which is the formula given in [11].

4. COMPARISON OF WAVELET TRANSFORMS

We will now compare the various wavelet transform based on the magnitude of the leading constant C_φ^-. One difficulty is that the general form (19) is rather impractical for dealing with wavelets whose Fourier transform is specified through an infinite product in Fourier domain (cf. (10)). It is therefore crucial to search for a simpler formula in terms of the refinement filter h.

4.1 Simplified bound calculation for the wavelet transform

In the case of a wavelet transform, the scaling function φ is the solution of the two scale relation (9). By using a recursive computation that takes advantage of this relation, one can derive a much simpler formula for the bound constant (cf. [12])

$$C_\varphi^- = \frac{|Q(-1)|\sqrt{A_\varphi(-1)}}{2^{L+1}\sqrt{4^L - 1}}, \tag{20}$$

where L is the order of the transform, $A_\varphi(z)$ is defined by (5), and where $Q(z)$ is the remainder in the factorization

$$H(z) = 2^{-L}(1+z)^L \cdot Q(z). \tag{21}$$

If the wavelet transform is orthogonal then $A_\varphi(z) = 1$, and the determination of the bound constant is straightforward. Otherwise, it necessary to determine $A_\varphi(z)$, the z-transform of the autocorrelation sequence $a_\varphi(k) = \langle\varphi(x), \varphi(x-k)\rangle$. In the case where $H(z)$ is FIR, this can be done rather easily through the solution of a linear system of equation (cf. [17]). In fact, $a_\varphi(k)$ is the eigenvector of the transition operator for the filter $\frac{1}{2}H(z)H(z^{-1})$ that corresponds to the eigenvalue one [6].

Table I provides a comparison of the bound constant for various wavelet transforms. Note that the results for splines are applicable for any type of spline wavelet transform (orthogonal Battle-Lemarié [18, 19], semi-orthogonal Chui-Wang / Unser-Aldroubi [20, 21, 22], and biorthogonal Cohen-Daubechies-Feauveau [23]). For a given order L, we see that the Daubechies wavelets have the worst performance; splines are by far the best. These results suggest that splines at half the resolution can provide as good an approximation as Daubechies wavelets at twice the rate. In general, the performance is better for the scaling functions that are the most regular.

TABLE 1 : RESCALED BOUND CONSTANT $A_L^- = C_\varphi^- \cdot L!$ FOR DIFFERENT WAVELET FAMILIES.

L	Daubechies	closest-to-linear phase	coiflets	spline	Deslauriers-Dubuc
1	0.2887	0.2887		0.2887	
2	0.2236	0.2236	0.2124	0.07454	0.07454
3	0.2988	0.2988		0.03450	
4	0.5557	0.5557	0.4953	0.02182	0.1871
5	1.316	1.316		0.01734	
6	3.779	3.779	3.231	0.01655	1.212
7	12.74	12.74		0.01844	
8	49.35	49.35	40.92	0.02347	15.06
9	215.8	215.8		0.03362	

4.2 Splines versus Daubechies wavelets

There are two families of wavelets for which the bound constant can be determined analytically. The first are the splines for which one gets (cf. [11])

$$C_{\text{L,splines}} = \sqrt{\sum_{k=1}^{\infty} \frac{2}{(2\pi k)^{2L}}} = \sqrt{\frac{|B_{2L}|}{(2L)!}}. \tag{22}$$

On the right hand side, we have identified B_{2L}, which is Bernouilli's number of degree $2L$ [24]. Remarkably, it is also possible to obtain an explicit form for the Daubechies wavelets even if those wavelets don't have a closed form solution. The result, which is derived in [17], is the following

$$C_{\text{Daubechies}} = 4^{-L} \sqrt{\frac{1}{1-4^{-L}} \binom{2L-1}{L}}. \tag{23}$$

Thanks to those equations, we can derive the following asymptotic result, a most unusual occurrence of the number π!

Theorem 4.1: Asymptotically as $L \to +\infty$, the approximation quality of Daubechies wavelets is the same than that of splines sampled at a π-*times coarser* rate, or, equivalently

$$\lim_{L\to\infty} \left(C_{\text{Daubechies}} / C_{\text{splines}} \right)^{1/L} = \pi. \tag{24}$$

Proof: This is only a sketch because we assume that (22) and (23) are given; the most difficult part is to obtain this latter equation (cf. [17]).To get the asymptotic form of the B-spline constant, we note that for L large, the predominant term in the infinite sum is the first one, so that

$$\lim_{L\to\infty} C_{\text{L,splines}} = \frac{\sqrt{2}}{(2\pi)^L}$$

On the Daubechies side, we use Stirling's formula to get the asymptotic form of the factorials. Thus, we obtain

$$\lim_{L\to+\infty} C_{L,\text{Daubechies}} = \frac{2^{-L}}{\sqrt[4]{4\pi L}}$$

Finally, we take the ratio and determine the limit

$$\lim_{L\to\infty}\left(C_{\text{Daubechies}} / C_{\text{splines}}\right)^{1/L} = \lim_{L\to\infty}\left(\frac{\pi^L}{\sqrt{2\sqrt[4]{4\pi L}}}\right)^{1/L} = \pi .$$

References

[1] G. Strang, "Wavelets and dilation equations: a brief introduction," *SIAM Review*, vol. 31, pp. 614-627, 1989.

[2] S.G. Mallat, "Multiresolution approximations and wavelet orthogonal bases of $L^2(R)$," *Trans. Amer. Math. Soc.*, vol. 315, no. 1, pp. 69-87, 1989.

[3] S. Mallat and W.L. Hwang, "Singularity detection and processing with wavelets," *IEEE Trans. Information Theory*, vol. 38, no. 2, pp. 617-643, March 1992.

[4] I. Daubechies, *Ten lectures on wavelets*. Philadelphia, PA: Society for Industrial and Applied Mathematics, 1992.

[5] S. Mallat, *A wavelet tour of signal processing*. San Diego: Academic Press, 1998.

[6] G. Strang and T. Nguyen, *Wavelets and filter banks*. Wellesley, MA: Wellesley-Cambridge, 1996.

[7] M. Vetterli and J. Kovacevic, *Wavelets and Subband Coding*. Englewood Cliffs, NJ: Prentice Hall, 1995.

[8] G. Strang and G. Fix, "A Fourier analysis of the finite element variational method," in *Constructive Aspect of Functional Analysis*, Rome: Edizioni Cremonese, pp. 796-830, 1971.

[9] W. Sweldens and R. Piessens, "Quadrature formulae and asymptotic error expansions for wavelet approximations of smooth functions," *SIAM J. Numer. Anal.*, vol. 31, no. 4, pp. 1240-1264, August 1994.

[10] W. Sweldens and R. Piessens, "Asymptotic error expansions for wavelet approximations of smooth functions II," *Numer. Math.*, vol. 68, no. 3, pp. 377-401, 1994.

[11] M. Unser, "Approximation power of biorthogonal wavelet expansions," *IEEE Trans. Signal Processing*, vol. 44, no. 3, pp. 519-527, March 1996.

[12] T. Blu and M. Unser, "Approximation error for quasi-interpolators and (multi-) wavelet expansions," *Applied and Computational Harmonic Analysis*, in press.

[13] A. Aldroubi and M. Unser, "Sampling procedures in function spaces and asymptotic equivalence with Shannon's sampling theory," *Numer. Funct. Anal. and Optimiz.*, vol. 15, no. 1&2, pp. 1-21, February 1994.

[14] G. Strang, "The finite element method and approximation theory," in *Numerical Solution of Partial Differential Equations - II*, B. Hubbard, Ed., New York: Academic Press, pp. 547-583, 1971.

[15] W.A. Light, "Recent developments in the Strang-Fix theory for approximation orders," in *Curves and Surfaces*, P.J. Laurent, A.L. Méhauté and L.L. Schumaker, Eds., Boston: Academic Press, pp. 285-292, 1991.

[16] T. Blu and M. Unser, "Quantitative Fourier analysis of approximation techniques: Part I—interpolators and projectors," *IEEE Transactions on Signal Processing*, submitted.

[17] T. Blu and M. Unser, "Quantitative Fourier analysis of approximation techniques: Part II—wavelets," *IEEE Transactions on Signal Processing*, submitted.

[18] G. Battle, "A block spin construction of ondelettes. Part I: Lemarié functions," *Commun. Math. Phys.*, vol. 110, pp. 601-615, 1987.

[19] P.-G. Lemarié, "Ondelettes à localisation exponentielles," *J. Math. pures et appl.*, vol. 67, no. 3, pp. 227-236, 1988.

[20] C.K. Chui and J.Z. Wang, "On compactly supported spline wavelets and a duality principle," *Trans. Amer. Math. Soc.*, vol. 330, no. 2, pp. 903-915, 1992.

[21] M. Unser, A. Aldroubi and M. Eden, "On the asymptotic convergence of B-spline wavelets to Gabor functions," *IEEE Trans. Information Theory*, vol. 38, no. 2, pp. 864-872, March 1992.

[22] M. Unser, A. Aldroubi and M. Eden, "A family of polynomial spline wavelet transforms," *Signal Processing*, vol. 30, no. 2, pp. 141-162, January 1993.

[23] A. Cohen, I. Daubechies and J.C. Feauveau, "Bi-orthogonal bases of compactly supported wavelets," *Comm. Pure Appl. Math.*, vol. 45, pp. 485-560, 1992.

[24] M. Abramowitz and I.A. Stegun, *Handbook of mathematical functions*. National Bureau of Standards, 1972.