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Complex B-splines

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Abstract

We propose a complex generalization of Schoenberg's cardinal splines. To this end, we go back to the Fourier domain definition of the B-splines and extend it to complex-valued degrees. We show that the resulting complex B-splines are piecewise modulated polynomials, and that they retain most of the important properties of the classical ones: smoothness, recurrence, and two-scale relations, Riesz basis generator, explicit formulae for derivatives, including fractional orders, etc. We also show that they generate multiresolution analyses of $L^2(\mathbb{R})$ and that they can yield wavelet bases. We characterize the decay of these functions which are no-longer compactly supported when the degree is not an integer. Finally, we prove that the complex B-splines converge to modulated Gaussians as their degree increases, and that they are asymptotically optimally localized in the time–frequency plane in the sense of Heisenberg's uncertainty principle.
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1. Introduction

Splines are very useful functions for mathematical analysis as well as for signal- and image processing, analysis, and representation, for computer graphics and many more [1–4]. The basis functions for I.J. Schoenberg's polynomial splines with uniform knots [5,6] are

$$\beta^n(t) = \frac{1}{n!}\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}(t-k)_+^n, \quad n \in \mathbb{N}.$$

Splines had their second breakthrough as G. Battle [7] and P.-G. Lemarié [8] discovered that B-splines generate multiresolution analyses and wavelets. Their simple form and compact support, in particular, was convenient for designing multiresolution algorithms and fast implementations. In [9], T. Blu and M. Unser gave an extension of B-splines to fractional orders. They showed that all the desirable properties of cardinal B-splines carry over to the

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fractional case. Moreover, fractional splines can be designed to have an arbitrary order of smoothness. They generate multiresolution analyses, and the FFT-based algorithm provides a fast method for signal analysis [10].

In this paper, we propose an analytical extension of the fractional B-spline approach that yields complex-valued functions. We perform this by complexifying the exponent; i.e., the order of the B-splines. In this way, we construct complex B-splines, which retain many properties of the classical real-valued B-splines; in particular, recurrence relations, smoothness and decay properties and multiresolution embeddings.

In the literature, various approaches have been proposed to extend polynomial splines to a complex setting. Depending on the method of construction, they can be classified into (1) complex curve splines and (2) planar splines.

J.H. Ahlberg, E.N. Nilson, and J.L. Walsh [11] were among the first to construct complex curve splines. They considered knots $t_1, \dots, t_N$ on a rectifiable Jordan curve K, and interpolated a function $f : K \to \mathbb{C}$ with complex cubics

$$q_\Delta(t) = a(t_j - t)^3 + b(t - t_{j+1})^3 + c(t_j - t) + d(t - t_{j+1})$$

for t on the arc K_j connecting t_{j-1} and t_j, such that $f(t_j) = q_\Delta(t_j)$ and $q'_\Delta(t_j) = q'_\Delta(t_{j+1})$. They also introduced an extension to the bounded domain interior of K via the Cauchy integral formula. However, extensions to higher degree, uniqueness and existence results for complex interpolating splines with equidistant and non-equidistant knots needed several decades of research to be completely settled. For references on results, we refer to [12, Ch. 1, §1. 7, Note 3].

H. Chen [12] considered the Torus $K = \mathbb{T}$ and defined complex B-splines via divided differences

$$N_{j,n}(z) = (z_{j+n+1} - z_j)[z_j, \dots, z_{j+n+1}]_s (s - z)_+^n,$$

where $(s - z)_+^l$ describes a polynomial of order l over some interval of $\mathbb{T}$. These splines satisfy a recursion formula and reproduce polynomials of degree n. Within this framework, Chen also defined complex harmonic splines via the Poisson integral formula. However, since these are periodic functions in $L^2(\mathbb{T})$, they cannot be seen as a direct extension of B-splines which live in $L^2(\mathbb{R})$.

G. Opfer and M.L. Puri [13] defined complex planar splines of the form $p(z) = \sum_{j,k=0}^{N} a_{j,k} z^j \bar{z}^k$, $a_{j,k} \in \mathbb{C}$, on triangulations of the complex plane. The monograph of G. Walz [14] gives an overview of complex splines on curves in the complex plane and of planar splines. Complex planar splines are of special relevance for the analysis of conformal mappings.

All these approaches and their approximation properties strongly depend on the choice of underlying bounded domains, meshes or rectifiable Jordan curves. Moreover, these complex splines have only been specified for integer degrees so far.

In this paper, we propose a natural extension of B-splines to complex splines on $\mathbb{R}$, which does not depend on the choice of certain curves or domains, and which is possible for all degrees $\alpha \in \mathbb{R}^+$. Similar to their real counterparts, our complex splines generate multiresolution analyses for $L^2(\mathbb{R})$. Moreover, the simple form of the scaling function in the Fourier domain allows the direct use of the Mallat algorithm [15] for wavelet analysis. This makes them easily accessible for applications.

This paper is organized as follows: We start with a short motivation for our construction. In fact, our complex B-splines are a generalization of fractional ones [9] to complex degrees. In the next section, we give a proper definition in Fourier domain and show that this construction is well defined. Section 4 is concerned with the time-domain representation, whereas Section 5 concentrates on B-spline properties, such as smoothness and decay, recurrence relations, and differential properties. We show that all those properties carry over smoothly from the fractional case.

In Section 6 we show that our construction of complex B-splines generates multiresolution analyses of $L^2(\mathbb{R})$. The respective refinement filters have a closed form and allow a fast implementation in Fourier domain.

In Section 7 and 8, we investigate the asymptotic behavior of complex B-splines. We show that they converge to Gabor functions as their degree increases and give an order of convergence. Moreover, we show that they approximately satisfy the lower bound of the Heisenberg uncertainty principle and thus converge to optimally time–frequency localized functions. Interestingly, the same results apply for cardinal B-splines and their fractional generalization. With our explicit order of convergence, we also contribute to the theory on those function families.

2. Turning classical B-splines to complex B-splines

Cardinal B-splines $\beta^n(x)$, here given in their Fourier domain representation

$$\hat{\beta}^n(\omega) = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^{n+1}, \quad n \in \mathbb{N},$$

have proved to be appropriate bases for many theoretical problems [3] as well as applications, including the life sciences [16,17]. However, they are piecewise polynomial functions and thus their order of smoothness—or, equivalently, their order of approximation—cannot be adjusted continuously. This problem was considered by T. Blu and M. Unser in [9] and solved by introducing fractional splines with a fractional exponent $\alpha \in \mathbb{R}$. They defined two versions of fractional B-splines: The causal one, β_+^α,

$$\hat{\beta}_+^\alpha(\omega) = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^{\alpha+1},$$

and the symmetric one, β_*^α, given by

$$\hat{\beta}_*^\alpha(\omega) = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^{\frac{\alpha+1}{2}} \left(\frac{1-e^{i\omega}}{-i\omega}\right)^{\frac{\alpha+1}{2}}.$$

Both types are in $L^1(\mathbb{R})$ if $\alpha > -1$ and in $L^2(\mathbb{R})$ if $\alpha > -\frac{1}{2}$. Later they introduced a further parameter $\tau \in \mathbb{R}$ describing shifts in the time domain [18,19]:

$$\hat{\beta}_\tau^\alpha(\omega) = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^{\frac{\alpha+1}{2}-\tau} \left(\frac{1-e^{i\omega}}{-i\omega}\right)^{\frac{\alpha+1}{2}+\tau}.$$

The B-splines are all scaling functions and can be used to specify dyadic multiresolution analyses of $L^2(\mathbb{R})$. However, they are real-valued. To define complex splines, we extend the construction one step further by considering complex-valued exponents instead of real ones.

3. Definition of complex B-splines

Given two complex numbers $z, w \neq 0$, we define the exponentiation operation as

$$w^z := e^{z(\ln|w| + i \arg w)}, \tag{1}$$

where the representation $w = |w|e^{i \arg w}$, with $\arg w \in [-\pi, \pi[$, is unique. This means that we only consider the principal branch of the complex exponential function. As usual $0^z = 0$ and $w^0 = 1$.

Definition 1. Suppose $z = \alpha + i\gamma \in \mathbb{C}$, $\alpha > -\frac{1}{2}$, $\gamma \in \mathbb{R}$. The complex B-spline β^z of complex degree z is defined in $L^2(\mathbb{R})$ via its Plancherel transform

$$\hat{\beta}^z(\omega) = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^{z+1}.$$

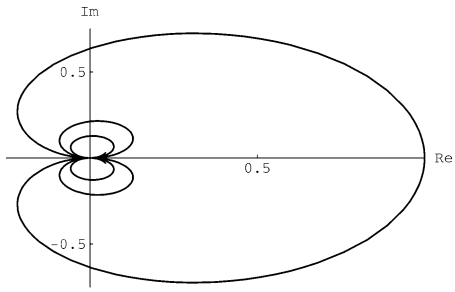
The shifted variant β_y^z is defined in $L^2(\mathbb{R})$ by

$$\hat{\beta}_y^z(\omega) = \hat{\beta}_{\tau+i\eta}^{\alpha+i\gamma}(\omega) = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^{\frac{z+1}{2}-y} \left(\frac{1-e^{i\omega}}{-i\omega}\right)^{\frac{z+1}{2}+y},$$

where $z = \alpha + i\gamma$, $y = \tau + i\eta$ with parameters $\alpha > -\frac{1}{2}$, $\gamma, \tau, \eta \in \mathbb{R}$.

Theorem 2. *The complex B-splines β^z and β_y^z are both well-defined, uniformly continuous, and elements of $L^2(\mathbb{R})$.*

Proof. Consider the function $\Omega(\omega) = \frac{1-e^{-i\omega}}{i\omega}$. Obviously, f has a continuation $\Omega(0) = 1$ and zeros at the points $\omega \in 2\pi\mathbb{Z} \setminus \{0\}$. The values of the function Ω never touches the negative real axis (see Fig. 1): $\operatorname{Im}\Omega(\omega) = \frac{1-\cos\omega}{i\omega} = 0$



$x \in [-20, 20]$

Fig. 1. The function $\Omega(\omega) = \frac{1-e^{-i\omega}}{i\omega}$ describes a curve in the complex plane, which never touches the negative real axis.

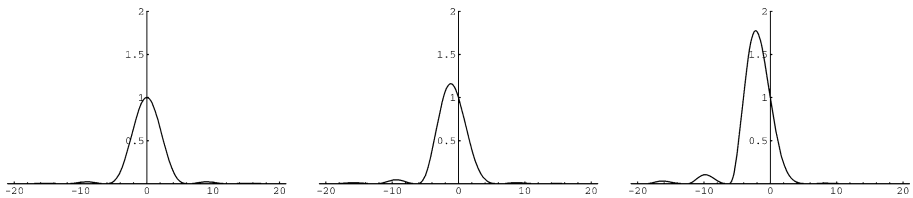


Fig. 2. The parameter η in the definition of the complex B-splines causes a one-sided enhancement of the frequency response of β_y^z. This appears as a shift of the functions' spectrum. Left: Symmetric fractional B-spline for $\alpha = 3$ and $\tau = y = 0$. Center and right: The complex shift $y = \tau + i\eta$ is increased to $\eta = 0.25$ and $\eta = 0.5$.

if and only if $\omega \in 2\pi\mathbb{Z}$, and $\operatorname{Re}\Omega(\omega) = \frac{\sin\omega}{\omega}$ is either zero or one at these points. We can always stick to the main branch when considering the complex exponents or complex logarithms. Thus $(\Omega(\omega))^z$ is uniquely defined according to (1). Hence $\hat{\beta}^z$ and $\hat{\beta}_y^z$ are well defined.

By exploiting the $L^2(\mathbb{R})$ inclusions of fractional B-splines [9], we get the following estimate:

$$\|\hat{\beta}^z\|_2^2 = \int_{\mathbb{R}} \left| \left(\frac{1-e^{-i\omega}}{i\omega} \right)^{\alpha+1} e^{-i\gamma \ln|\Omega(\omega)|} e^{\gamma \arg \Omega(\omega)} \right|^2 \mathrm{d}\omega \leqslant e^{2\gamma\pi} \int_{\mathbb{R}} \left| \left(\frac{1-e^{-i\omega}}{i\omega} \right)^{\alpha+1} \right|^2 \mathrm{d}\omega = e^{2\gamma\pi} \|\hat{\beta}_+^\alpha\|_2^2 < \infty.$$

Analogously,

$$\|\hat{\beta}_y^z\|_2^2 = \int_{\mathbb{R}} \left| \hat{\beta}_\tau^\alpha(\omega) e^{2\eta \arg \Omega(\omega)} \right|^2 \mathrm{d}\omega \leqslant e^{2\eta\pi} \|\hat{\beta}_\tau^\alpha\|_2^2 < \infty.$$

Thus, both functions belong to $L^2(\mathbb{R})$ and their Plancherel inverses exist. Hence $\beta^z, \beta_y^z \in L^2(\mathbb{R})$.

In the same way, one can show that $\hat{\beta}^z$ and $\hat{\beta}_y^z$ are both elements of $L^1(\mathbb{R})$ for $\operatorname{Re} z > -1$. Hence, β^z and β_y^z are uniformly continuous. □

When compared to real-valued fractional B-splines β_+^α and β_τ^α, the imaginary part $\operatorname{Im} z = \gamma$ of the complex exponent z and the complex shift y have the following effects:

For $\hat{\beta}^z(\omega) = \hat{\beta}_+^\alpha(\omega) e^{-i\gamma \ln|\Omega(\omega)|} e^{\gamma \arg \Omega(\omega)}$, the parameter γ introduces a phase and a scaling factor. In fact, the frequency components on the negative and positive real axis are enhanced with different sign, since $\arg \Omega(\omega) \geqslant 0$ for $\omega \leqslant 0$ and $\arg \Omega(\omega) \leqslant 0$ otherwise. This has the effect of shifting the frequency spectrum towards the negative or positive frequency side, depending on the sign of γ.

For the shifted complex B-spline $\hat{\beta}_y^z(\omega) = \hat{\beta}_\tau^\alpha(\omega) e^{i\gamma \ln|\Omega(\omega)|} e^{2\eta \arg \Omega(\omega)}$, the complex exponent γ also introduces a phase factor. The scaling factor here involves η, which influences the enhancement of positive (resp., negative) frequency components analogously. Figure 2 illustrates this effect. Note that $\hat{\beta}_{i\eta}^\alpha$ is a real-valued function for all $\eta \in \mathbb{R}$.

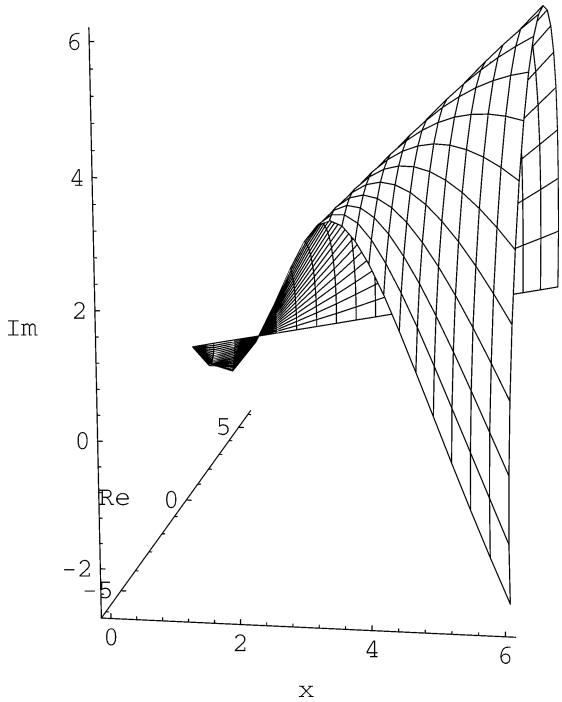


Fig. 3. Complex truncated power function x_+^z for $z = 1 + i\gamma$, $\gamma \in [0, 2]$. The straight line in the background represents the power function x_+.

4. Time domain representation

It is well known that B-splines and fractional B-splines can be represented as a series of truncated power functions. The same can be proved for complex B-splines.

4.1. Time domain representation for β^z

Let x_+^z denote the truncated power function of complex degree z with knot zero:

$$x_+^z = \begin{cases} x^z = e^{z\cdot\ln x} = x^{\operatorname{Re} z} e^{i \operatorname{Im} z \ln x} & \text{for } x > 0, \\ 0 & \text{elsewhere.} \end{cases}$$

For an example, see Fig. 3. Obviously $|x_+^z| = x_+^{\operatorname{Re} z}$. The imaginary part introduces a phase factor.

For $\operatorname{Re} z > 0$, $\operatorname{Re} z \notin \mathbb{N}$ and $\operatorname{Im} z \neq 0$, we have with the following distributional Fourier transform [20]

$$\left(x_+^z\right)^{\hat{}}(\omega) = \frac{1}{(i\omega)^{z+1}}\Gamma(z+1),$$

where the last but one equation can be derived using Cauchy's integral theorem for holomorphic functions. If n is a positive integer, we have

$$\left(x_+^n\right)^{\hat{}}(\omega) = \frac{\Gamma(n+1)}{(i\omega)^{n+1}} + i^n \pi \delta^{(n)}(\omega).$$

Here Γ denotes Euler's Gamma function, which is defined on the set $\mathbb{C} \setminus \{0, -1, -2, -3, \ldots\}$.

The truncated power function x_+^z and their integer shifts $(x-k)_+^z$, $k \in \mathbb{Z}$, are the basic atoms for the time domain representation of β^z.

Theorem 3. *The time domain representation of the complex B-spline β^z is*

$$\beta^z(x) = \frac{1}{\Gamma(z+1)} \sum_{k \geqslant 0} (-1)^k \binom{z+1}{k} (x-k)_+^z. \tag{2}$$

This equation is valid pointwise for all $x \in \mathbb{R}$ and also in the $L^2(\mathbb{R})$-sense.

Before proving this theorem, we have a closer look at the differences on the right-hand side of (2).

For fixed $z \in \mathbb{C}$ this is a causal complex difference operator

$$C_c(\mathbb{R}) \to C_c(\mathbb{R}), \qquad f \mapsto \Delta^z f = \sum_{k \geqslant 0} (-1)^k \binom{z}{k} f(\bullet - k).$$

It satisfies $\Delta^z(xf(x)) = (x - z)\Delta^z f(x) + z\Delta^{z-1} f(x)$, and for all $z, w \in \mathbb{C}$ and for compactly supported continuous functions $f, g \in C_c(\mathbb{R})$, $\Delta^{z+w}(f * g) = \Delta^z f * \Delta^w g$, and thus extends the real-valued version in [9].

Proof of Theorem 3. For $\operatorname{Re} z > 0$, we consider the pointwise-defined function

$$\frac{1}{\Gamma(z+1)} \Delta^{z+1} x_+^z := \frac{1}{\Gamma(z+1)} \sum_{k \geqslant 0} (-1)^k \binom{z+1}{k} (x-k)_+^z.$$

For $z \notin \mathbb{N}$, its Fourier transform in the sense of tempered distributions is given by

$$\begin{aligned}
\frac{1}{\Gamma(z+1)} \big(\Delta^{z+1} x_+^z\big)^{\wedge}(\omega) &= \frac{1}{\Gamma(z+1)} \sum_{k \geqslant 0} (-1)^k \binom{z+1}{k} \int_{\mathbb{R}} (x-k)_+^z e^{-i\omega x}\, \mathrm{d}x \\
&= \frac{1}{\Gamma(z+1)} \sum_{k \geqslant 0} (-1)^k \binom{z+1}{k} \int_{\mathbb{R}} x_+^z e^{-i\omega(x+k)}\, \mathrm{d}x = \sum_{k \geqslant 0} (-1)^k \binom{z+1}{k} \frac{e^{-i\omega k}}{(i\omega)^{z+1}} \\
&= \frac{1}{(i\omega)^{z+1}} \big(1 - e^{-i\omega}\big)^{z+1} = \hat{\beta}^z(\omega).
\end{aligned}$$

Here, we used the dominated convergence theorem together with the fact that $\sum_{k \geqslant 0} \left|\binom{z+1}{k}\right| < \infty$. This can be seen from

$$\binom{z+1}{k} = \frac{\Gamma(z+2)}{\Gamma(k+1)\Gamma(z+2-k)} = \frac{1}{\Gamma(k+1)} z^k \big(1 + \mathcal{O}(1/z)\big)$$

for $z \to \infty$, along any curve joining $z = 0$ and $z = \infty$, since there $z^{b-a}\Gamma(z+a)/\Gamma(z+b) = 1 + \mathcal{O}(1/z)$, for $a, b > 0$. Thus,

$$\sum_{k \geqslant 0} \left| \binom{z+1}{k} \right| = \sum_{k \geqslant 0} \left| \frac{1}{\Gamma(k+1)} z^k \big(1 + \mathcal{O}(1/z)\big) \right| \leqslant \operatorname{const} e^{|z|}.$$

Hence, the Fourier transform of $\frac{1}{\Gamma(z+1)} \Delta^{z+1} x_+^z$ is the function $\hat{\beta}^z$. From a density argument, we deduce the same for $L^2(\mathbb{R})$- and $L^1(\mathbb{R})$-topology. Thus $\beta^z(x) = \frac{1}{\Gamma(z+1)} \Delta^{z+1} x_+^z$ is the time-domain representation of the complex B-spline, and converges pointwise.

For $z = N \in \mathbb{N}$, we recover Schoenberg's polynomial B-splines

$$\beta^N(x) = \frac{1}{\Gamma(N+1)} \Delta^{N+1} x_+^N = \frac{1}{\Gamma(N+1)} \sum_{k=0}^{N+1} (-1)^k \binom{N+1}{k} (x-k)_+^N$$

and

$$\hat{\beta}^N(\omega) = \left(\frac{1 - e^{-i\omega}}{i\omega} \right)^{N+1}.$$

This concludes the proof. □

This theorem shows that complex B-splines can be derived—as in the classical and in the fractional case—by applying a difference operator on the function $x \mapsto x_+^z$, $\operatorname{Re} z > 0$. Figure 4 gives some examples of this new spline family.

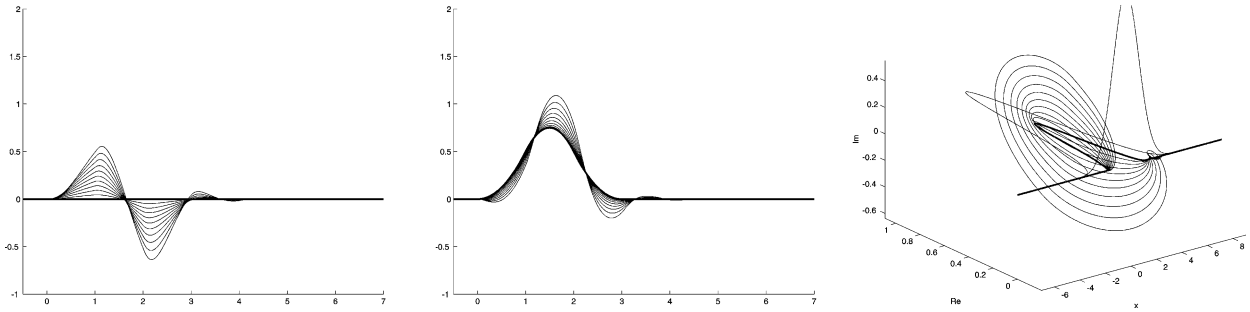


Fig. 4. Complex B-splines β^z for $z = 2 + ik0.1$, $k = 0, \ldots, 10$. Left: Real parts. Middle: Imaginary parts. Right: 3D view. The dashed lines show the Gaussian envelope.

4.2. Time domain representation of β_y^z

The same type of considerations also apply to β_y^z. In the following, our aim is to define an appropriate difference operator Δ_y^z and find a function ρ_y^z such that $\beta_y^z = \Delta_y^z \rho_y^z$.

Formally, we can factor $\hat{\beta}_y^z$ into a product of a 2π-periodic and a decaying component:

$$\hat{\beta}_y^z(\omega) = \left[\left(1 - e^{-i\omega}\right)^{\frac{z+1}{2} - y}\left(1 - e^{i\omega}\right)^{\frac{z+1}{2} + y}\right] \cdot \left[(i\omega)^{-\frac{z+1}{2} + y}(-i\omega)^{-\frac{z+1}{2} - y}\right]. \tag{3}$$

The 2π-periodic part corresponds to the difference operator, while the right-hand factor will yield the function ρ_y^z. We first consider the periodic part:

Lemma 4. *Suppose* $\operatorname{Re} u, \operatorname{Re} v > 0$. *Then*

$$(1 + z)^u \left(1 + z^{-1}\right)^v = \sum_{n \in \mathbb{Z}} \binom{u + v}{u - n} z^n,$$

where the right-hand side converges absolutely. It also converges in $L^2(\mathbb{T})$*, provided that* $\operatorname{Re} u + \operatorname{Re} v > -\frac{1}{2}$.

Proof. For $\operatorname{Re} u > 0$, the series

$$(1 + z)^u = \sum_{n=0}^{\infty} \binom{u}{n} z^n$$

converges absolutely for all $|z| \leqslant 1$. Thus, for $\operatorname{Re} v > 0$

$$\left(1 + z^{-1}\right)^v = \sum_{n=0}^{\infty} \binom{v}{n} z^{-n}$$

converges absolutely for all $|z| \geqslant 1$ [21, Satz 247, p. 440]. Hence their product converges absolutely for $|z| = 1$, and we can sum up using any ordering:

$$(1 + z)^u \left(1 + z^{-1}\right)^v = \sum_{n=0}^{\infty} \binom{u}{n} z^n \cdot \sum_{n=0}^{\infty} \binom{v}{n} z^{-n} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{u}{k + n} \binom{v}{k} z^n + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \binom{u}{k} \binom{v}{k + n} z^{-n}. \tag{4}$$

It is shown in [22, Theorems 2 and 8] that, for x, y with $\operatorname{Re}(x), \operatorname{Re}(y), \operatorname{Re}(x + y) > -1$ and $a \in \mathbb{C}$, there holds a generalized Vandermonde–Chu convolution formula

$$\sum_{k=0}^{\infty} \binom{x}{k} \binom{y}{a - k} = \binom{x + y}{a}. \tag{5}$$

The latter series is uniformly convergent on compact subsets with respect to a. This yields

$$\begin{aligned}(1+z)^u\left(1+z^{-1}\right)^v &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{u}{u-n-k}\binom{v}{k}z^n + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\binom{u}{k}\binom{v}{v-k-n}z^{-n}\\ &= \sum_{n=0}^{\infty}\binom{u+v}{u-n}z^n + \sum_{n=1}^{\infty}\binom{u+v}{v-n}z^{-n}\\ &= \sum_{n=0}^{\infty}\binom{u+v}{u-n}z^n + \sum_{n=1}^{\infty}\binom{u+v}{u+n}z^{-n} = \sum_{n\in\mathbb{Z}}\binom{u+v}{u-n}z^n.\end{aligned}$$

For $\operatorname{Re} u, \operatorname{Re} v > -1$, and $\operatorname{Re} u + \operatorname{Re} v > -\frac{1}{2}$, the formula is true in the L^2-sense, since the monomials $\{z^n,\ n \in \mathbb{Z}\}$ form an orthonormal basis of $L^2(\mathbb{T})$ and thus allow unconditional summation. □

Note 1. This formula was proved in [19] for real u, v using Cauchy's integral formula and a recurrence relation for $\binom{u+v}{u-n}$.

Now, we are able to define the difference operator $\Delta_y^z = \Delta_{\tau+i\eta}^{\alpha+i\gamma}$ by its Fourier or Plancherel transform as

$$\left(\Delta_y^z f\right)\hat{}\,(\omega) = \left(1-e^{i\omega}\right)^{\frac{z}{2}-y}\left(1-e^{-i\omega}\right)^{\frac{z}{2}+y}\hat{f}(\omega) = \sum_{n\in\mathbb{Z}}\binom{z}{\frac{z}{2}-y-n}(-1)^n e^{i\omega n}\hat{f}(\omega) \tag{6}$$

for $f \in L^2(\mathbb{R})$, or in the sense of tempered distributions, for $f \in \mathcal{S}'(\mathbb{R})$. Let

$$\left|\begin{matrix} z \\ k-y \end{matrix}\right| := \binom{z}{\frac{z}{2}+k-y}$$

denote the modified binomial coefficient. Then

$$\Delta_y^z f = \sum_{k\in\mathbb{Z}}(-1)^k\left|\begin{matrix} z \\ k-y \end{matrix}\right| f(\bullet - k) \tag{7}$$

for $f \in \mathcal{S}'$, $\operatorname{Re} z > 0$ and $y \in \mathbb{C}$.

For bandlimited functions f with $\operatorname{supp}\hat{f} \in [0, 2\pi]$, (6) simplifies to

$$\left(\Delta_y^z f\right)\hat{}\,(\omega) = \left(-ie^{i\frac{\omega}{2}}2\sin\frac{\omega}{2}\right)^{\frac{z}{2}-y}\left(ie^{-i\frac{\omega}{2}}2\sin\frac{\omega}{2}\right)^{\frac{z}{2}+y}\hat{f}(\omega) = \left|\sin\frac{\omega}{2}\right|^{i\gamma}e^{-i\tau(\omega+\pi)}e^{\eta\omega}e^{-\eta\pi}\left(\Delta_0^z f\right)\hat{}\,(\omega)$$

for $\omega \in [0, 2\pi]$, since $\sin\frac{\omega}{2} \geqslant 0$. Likewise, for f such that $\operatorname{supp}\hat{f} \in [-2\pi, 0]$, we get $(\Delta_y^z f)\hat{}\,(\omega) = |\sin\frac{\omega}{2}|^{i\gamma}e^{-i\tau\omega}e^{i\tau\pi} \times e^{\eta\omega}e^{\eta\pi}(\Delta_0^\alpha f)\hat{}\,(\omega)$. This yields the following interpretation: Frequency parts in $[0, 2\pi] + 4\pi\mathbb{Z}$ are enhanced by a factor $e^{-\eta\pi}$, whereas frequency parts in $[-2\pi, 0] + 4\pi\mathbb{Z}$ are enhanced by a factor $e^{\eta\pi}$. Thus, the global effect is that of a fractional difference with an enhancement (respectively attenuation) of the positive frequency components by $e^{-\eta\pi}$ and an attenuation of the negative components by $e^{\eta\pi}$.

Note that, for $\gamma = 0$, in time-domain $\Delta_y^\alpha f(t) = e^{-i\bar{y}\pi}\Delta_0^\alpha f(t+y)$ for f with $\operatorname{supp}\hat{f} \in [0, 2\pi]$, and similarly, $\Delta_y^\alpha f(t) = e^{-iy\pi}\Delta_0^\alpha f(t+\bar{y})$ for f with $\operatorname{supp}\hat{f} \in [-2\pi, 0]$. Thus, the parameter y acts as a combination of shift into the complex plane and modulation within the complex plane.

Considering (3), we now have to determine the inverse Fourier transform of

$$\hat{\rho}_y^z(\omega) = (-i\omega)^{-\frac{z+1}{2}-y}(i\omega)^{-\frac{z+1}{2}+y},$$

which will then finally yield the time-domain formula for β_y^z.

Theorem 5. *The complex B-spline β_y^z has the time domain representation*

$$\beta_y^z(x) = \Delta_y^z\rho_y^z(x) = \sum_{k\in\mathbb{Z}}(-1)^k\left|\begin{matrix} z+1 \\ k-y \end{matrix}\right|\rho_y^z(x-k), \tag{8}$$

with

$$\rho_y^z(t) = c_1(z)\frac{\cos \pi y}{\Gamma(z+1)}|t|^z + c_2(z)\frac{\sin \pi y}{\Gamma(z+1)}|t|^z \operatorname{sign} t,$$

where $c_1 = c_1(z) = -\frac{1}{2\sin\frac{\pi}{2}z}$ *and* $c_2 = c_2(z) = -\frac{1}{2\cos\frac{\pi}{2}z}$ *for* $z \notin \mathbb{N}_0$.

If $z \in \mathbb{N}_0$*, we have to take into account an supplementary logarithmic factor* $\ln|t|$*. In fact,*

$$c_1(2n,t) = (-1)^{n+1}\frac{1}{\pi}\ln|t| = c_2(2n+1,t) \quad \textit{and} \quad c_2(2n) = (-1)^{n+1}\frac{1}{2} = c_1(2n+1),$$

for $n \in \mathbb{N}_0$.

For $\operatorname{Re} z > 0$*, the series* (8) *converges in* $\mathcal{S}'(\mathbb{R})$.

Note 2. The translation-invariant space generated by shifted complex-valued B-splines $\beta_{i\eta}^z$ is included in the one generated by two real-shifted B-splines:

$$\beta_{i\eta}^z \in \overline{\operatorname{span}\{\beta_0^z(\bullet - k), \beta_{1/2}^z(\bullet - k);\ k \in \mathbb{Z}\}}^{L^2(\mathbb{R})}.$$

This can be seen from the fact that

$$\rho_{i\eta}^z = \cosh(\eta)\rho_0^z - i\sinh(\eta)\rho_{1/2}^z,$$

and from the estimates $\left|{z+1 \atop k-i\tau}\right| = \left|{z+1 \atop k}\right|\frac{2}{z^2}(1+\mathcal{O}(1/z))$ and $\left|{z+1 \atop k-i\tau}\right| = \left|{z+1 \atop k+\frac{1}{2}}\right|\frac{2}{z^2}(1+\mathcal{O}(1/z))$ for $z \to \infty$.

Proof. For almost all $\omega \in \mathbb{R}$, $a \in \mathbb{R}$, and $b \in \mathbb{C}$, the following is true:

$$(-i\omega)^{\frac{a}{2}-b}(i\omega)^{\frac{a}{2}+b} = \cos(\pi b)|\omega|^a + \sin(\pi b)i|\omega|^a \operatorname{sign}(\omega).$$

Substituting a by $-(z+1)$ and b by y yields

$$\begin{aligned}\hat{\beta}_y^z(\omega) &= \sum_{n\in\mathbb{Z}} \binom{z+1}{\frac{z+1}{2}+y-n}(-1)^n e^{in\omega}\hat{\rho}_y^z(\omega)\\ &= \sum_{n\in\mathbb{Z}} \binom{z+1}{\frac{z+1}{2}+y-n}(-1)^n e^{in\omega}\big(\cos(\pi y)|\omega|^{-(z+1)} + \sin(\pi y)i|\omega|^{-(z+1)}\operatorname{sign}\omega\big).\end{aligned}$$

Using the Fourier relations for generalized functions [20]

$$\mathcal{F}^{-1}\left(\frac{1}{|\bullet|^{z+1}}\right)(t) = \begin{cases} -\frac{|t|^z}{2\Gamma(z+1)\sin\frac{\pi}{2}z} & \text{if } \operatorname{Re} z > -1 \text{ and } z \notin 2\mathbb{N},\\ (-1)^{\frac{z}{2}+1}\frac{t^z\ln|t|}{\pi\Gamma(z+1)} & \text{if } z \in 2\mathbb{N},\end{cases} \tag{9}$$

and

$$\mathcal{F}^{-1}\left(i\frac{\operatorname{sign}\bullet}{|\bullet|^{z+1}}\right)(t) = \begin{cases} -\frac{|t|^z\operatorname{sign} t}{2\Gamma(z+1)\cos\frac{\pi}{2}z} & \text{if } \operatorname{Re} z > -1 \text{ and } z \notin 2\mathbb{N}-1,\\ (-1)^{\frac{z-1}{2}}\frac{t^z\ln|t|}{\pi\Gamma(z+1)} & \text{if } z \in 2\mathbb{N}-1,\end{cases} \tag{10}$$

yields the explicit formula

$$\beta_y^z(x) = \Delta_y^{z+1}\rho_y^z(x) = \sum_{k\in\mathbb{Z}}(-1)^k\left|{z+1 \atop k-y}\right|\rho_y^z(x-k),$$

with ρ_y^z as above. □

This proves that the representation given in [19] for real exponents α extends to the complex case as well. Moreover, for non-integer α, the function ρ_z is selfsimilar; i.e.,

$$\rho_y^z(\lambda x) = \lambda^z\rho_y^z(x)$$

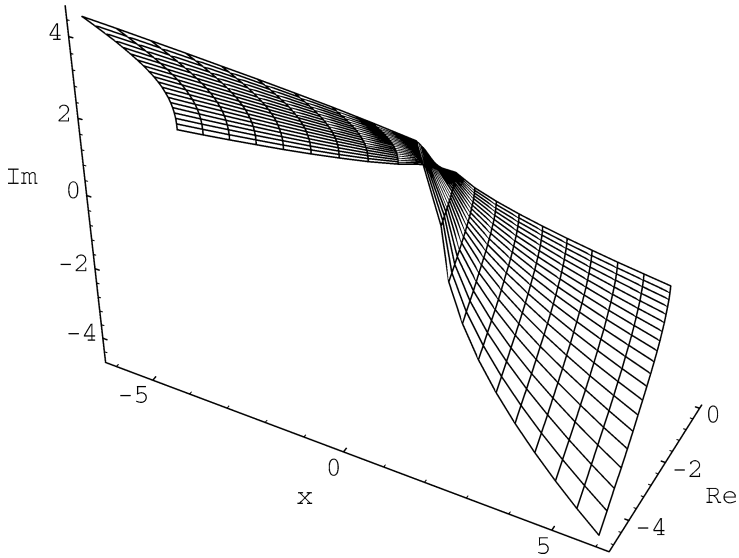


Fig. 5. Central basic function ρ_y^z for $z = \alpha = 0.5$, $y = i\eta$, $\eta \in [0, 0.5]$.

for positive λ. Thus ρ_y^z is a complex extension of the admissible central basis functions defined in [23], and $\beta_y^z(x)$ is the localization of the complex central basis function ρ_y^z.

From the same viewpoint, β^z is the localization of the self-similar central basis function x_+^z, since for all $\lambda > 0$ it is $(\lambda x)_+^z = \lambda^z x_+^z$.

Examples of the complex splines β_*^z are given in Fig. 6. Note that for $z = \alpha \in \mathbb{R}^+$ the real part is an even function and the imaginary part odd. This is due to the fact that $\hat{\beta}_y^\alpha$ is real-valued.

5. Basic properties

5.1. Continuity and decay

Since $\hat{\beta}^z$ and $\hat{\beta}_y^z$ are both elements of $L^2(\mathbb{R})$ for $\alpha > -\frac{1}{2}$, continuous, and decaying like $\mathcal{O}\big(\frac{1}{|\omega|^{\alpha+1}}\big)$ as $|\omega| \to \infty$, we deduce that the complex B-splines belong to the following Sobolev spaces:

$$\beta^z, \beta_y^z \in W_2^r(\mathbb{R}) \quad \text{for } r < \alpha + \frac{1}{2}.$$

The order of zeros at the points $\omega = 2\pi k$, $k \neq 0$, is $\alpha + 1$, which can be seen from

$$\hat{\beta}_y^z(\omega + 2\pi k) = \hat{\beta}_y^z(\omega)\left(\frac{\omega}{\omega + 2\pi k}\right)^{\frac{z+1}{2} - y}\left(\frac{\omega}{\omega + 2\pi k}\right)^{\frac{z+1}{2} + y} = \hat{\beta}_y^z(\omega)\mathcal{O}\big(\|\omega\|^{z+1}\big) \quad \text{for } \omega \to 0.$$

The same is true for $\hat{\beta}^z$. Thus, $D^m\hat{\beta}^z$ is continuous for $m < \alpha + 1$ and $D^m\hat{\beta}^z \in L^p(\mathbb{R})$ for $1 \leqslant p \leqslant \infty$ and $p(\alpha + 1 - m) > -1$; i.e., $m < \alpha + 1 + \frac{1}{p}$.

Considering $p = 1$, $x^m\beta^z$ is uniformly continuous for $m < \alpha + 2$ and vanishing at infinity. Thus $\beta^z(x) = \mathcal{O}(x^m)$ for $|x| \to \infty$ and $m < \alpha + 2$. The same reasoning applies for β_y^z.

Note 3. The complex B-spline regularity as well as the decay only depend on the real part $\operatorname{Re} z = \alpha$ of the complex exponent z. On the other hand, the imaginary part $\operatorname{Im} z = \gamma$ influences the damping (resp., enhancement) of frequencies (cf. Section 3).

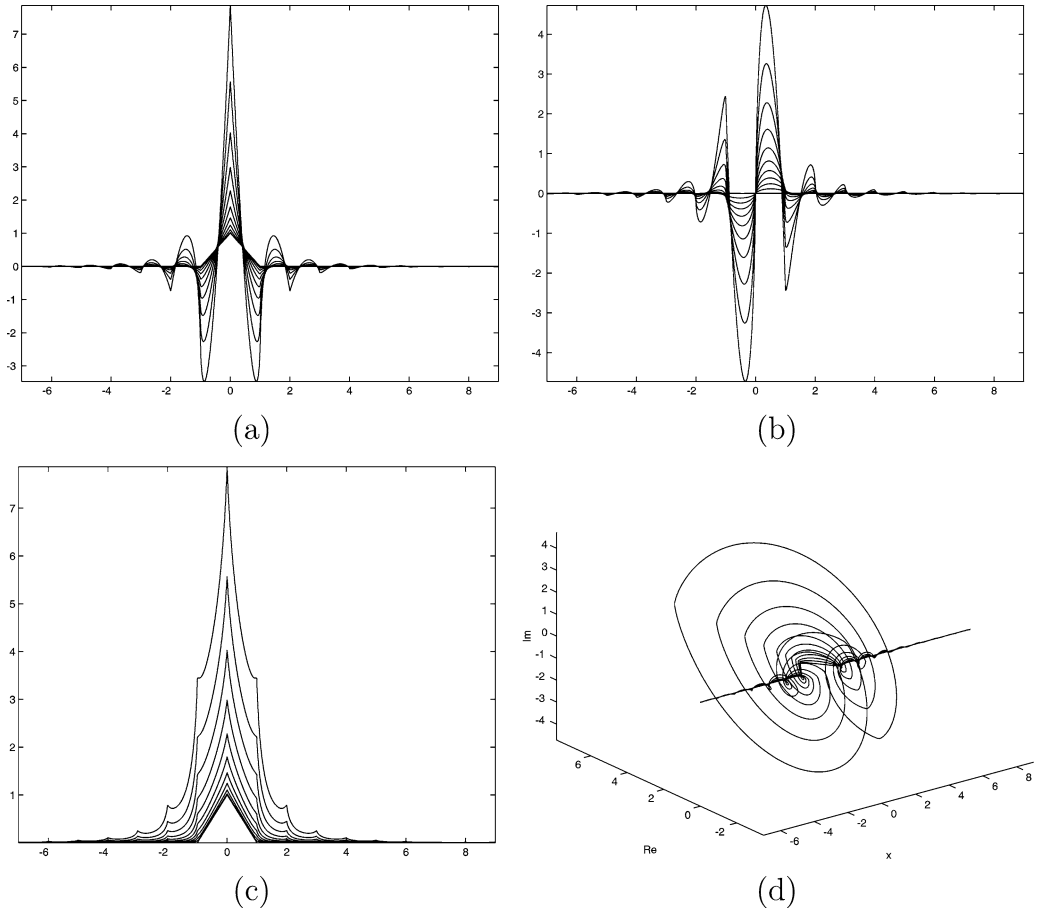


Fig. 6. The splines β_y^z for $z = \alpha = 1$ and $y = i\eta$, $\eta = 0, 0.1, \ldots, 1$. (a) Real part, (b) imaginary part, (c) absolute value. (d) shows the splines in a 3D view with respect to real and imaginary part. The thicker triangles in (a), (c) and (d) correspond to the real-valued piecewise linear B-spline β^1.

5.2. *Recurrence relations and differential properties*

The classical B-splines satisfy some well-known recurrence relations, such as iterative construction by a recursion formula, and convolution equations. In [9], it is shown that these carry over to fractional splines. The same is true for complex splines:

Proposition 6. *The complex fractional B-spline* β^z, $\operatorname{Re} z = \alpha > 0$, *satisfies*

(i) $\beta^z(x) = \frac{x}{z}\beta^{z-1}(x) + \frac{z+1-x}{z}\beta^{z-1}(x-1)$.
(ii) $\beta^{z_1} * \beta^{z_2} = \beta^{z_1+z_2+1}$.
(iii) $D^{z_1}\beta^z = \Delta_+^{z_1}\beta^{z-z_1}$ *for all* $\operatorname{Re} z_1 > \operatorname{Re} z$. *Here, the differential operator* D^{z_1} *is defined on* $\mathcal{S}(\mathbb{R})$ *via its Fourier transform* $(D^{z_1} f)\hat{}(\omega) = (i\omega)^{z_1}\hat{f}(\omega)$.

Proof. The proofs in [9] can be directly adapted for complex exponents, since both Gamma and Beta function have analytic continuations to the complex halfplane $\operatorname{Re} z > 0$. □

The complex B-splines β_y^z satisfy a recurrence formula relating the B-splines of degree z and $z-2$. This is due their Definition 1, where the complex degree z splits in $z + 1/2$. Moreover, the corresponding difference operator Δ_y^z behaves as a weighted sum of fractional derivatives:

Proposition 7. (i) *The shifted complex B-spline* $\beta_y^z = \beta_{\tau+i\eta}^{\alpha+i\gamma}$ *satisfies the following recurrence relation*:

$$\beta_y^z(x) = \frac{\left(x-y+\frac{z+1}{2}\right)^2}{z(z-1)}\beta_y^{z-2}(x+1) - 2\frac{\left((x-y)^2+\frac{1-z^2}{4}\right)}{z(z-1)}\beta_y^{z-2}(x) + \frac{\left(x-y-\frac{z+1}{2}\right)^2}{z(z-1)}\beta_y^{z-2}(x-1) \tag{11}$$

for all $\operatorname{Re} z = \alpha > 1$ *and all* $y \in \mathbb{C}$.

(ii) $\beta_{y_1}^{z_1} * \beta_{y_2}^{z_2}(t) = \beta_{y_1+y_2}^{z_1+z_2+1}(t)$.

(iii) *Explicit differentiation formula*:

$$\Delta_y^z \beta_{y_1}^{z_1} = \partial_y^z \beta_{y_1+y}^{z_1+z}, \tag{12}$$

where ∂_y^z *is a fractional differential operator, defined in the sense of tempered distributions*:

$$\partial_y^z f(t) = \frac{\Gamma(z+1)}{2\pi}\left(\kappa(z,y)\,{}_{-\infty}\mathbf{D}_t^z f(t) + \kappa(z,-y)\,{}_{-\infty}\mathbf{D}_{-t}^z f(-t)\right),$$

with $\kappa(z,y) = e^{-iy\pi}e^{-i(z+1)\frac{\pi}{2}} + e^{iy\pi}e^{i(z+1)\frac{\pi}{2}}$, *and*

$${}_{-\infty}\mathbf{D}_t^z f(t) = \int_{-\infty}^{t} \frac{f(s)}{(t-s)^{z+1}}\,\mathrm{d}s. \tag{13}$$

Proof. The proof of (i) is a direct extension of the one of [9, Proposition 2.6] for real-valued fractional B-splines.

From formula (6), we easily see that $\Delta_{y_1}^{z_1}\Delta_{y_2}^{z_2} f(x) = \Delta_{y_1+y_2}^{z_1+z_2} f(x)$. Taking the Fourier transform yields (ii).

In order to verify (iii), we consider the Fourier domain version of the right side of Eq. (12):

$$\begin{aligned}
\left(\Delta_y^z \beta_{y_1}^{z_1}\right)^{\wedge}(\omega) &= \left(1-e^{i\omega}\right)^{\frac{z}{2}-y}\left(1-e^{-i\omega}\right)^{\frac{z}{2}+y}\hat{\beta}_{y_1}^{z_1}(\omega) \\
&= (-i\omega)^{\frac{z}{2}-y}(i\omega)^{\frac{z}{2}+y}\left(\frac{1-e^{-i\omega}}{i\omega}\right)^{\frac{z_1+z+1}{2}-y_1+y}\left(\frac{1-e^{i\omega}}{-i\omega}\right)^{\frac{z_1+z+1}{2}+y_1-y} \\
&= (-i\omega)^{\frac{z}{2}-y}(i\omega)^{\frac{z}{2}+y}\hat{\beta}_{y_1-y}^{z_1+z}.
\end{aligned} \tag{14}$$

Next, we simplify the first two factors: $(i\omega)^{\frac{z}{2}+y}(-i\omega)^{\frac{z}{2}-y} = |\omega|^z e^{iy\pi\,\mathrm{sign}(\omega)}$. We then calculate the generalized inverse Fourier transform:

$$\begin{aligned}
\mathcal{F}^{-1}\left(|\omega|^z e^{iy\pi\,\mathrm{sign}(\omega)}\right)(t) &= \frac{e^{-iy\pi}}{2\pi}\int_{\mathbb{R}} \omega_+^z e^{-i\omega t}\,\mathrm{d}\omega + \frac{e^{iy\pi}}{2\pi}\int_{\mathbb{R}} \omega_+^z e^{-i\omega(-t)}\,\mathrm{d}\omega = \frac{\Gamma(z+1)}{2\pi}\left(\frac{e^{-iy\pi}}{(it)^{z+1}} + \frac{e^{iy\pi}}{(-it)^{z+1}}\right) \\
&= \frac{\Gamma(z+1)}{2\pi|t|^{z+1}}\left(e^{-iy\pi}e^{-(z+1)i\frac{\pi}{2}\,\mathrm{sign}(t)} + e^{iy\pi}e^{(z+1)i\frac{\pi}{2}\,\mathrm{sign}(t)}\right),
\end{aligned}$$

which holds as long as $\operatorname{Re} z > 0$ and $z \notin \mathbb{N}$. In the case $z = n \in \mathbb{N}$, the term $\frac{i^n}{2}(e^{iy\pi}\delta^{(n)}(-t) + e^{-iy\pi}\delta^{(n)}(t))$ has to be added to the result.

Since multiplication in Fourier domain is a convolution in time domain, we get

$$\begin{aligned}
\Delta_y^z \beta_{y_1}^{z_1}(t) &= \frac{\Gamma(z+1)}{2\pi}\int_{\mathbb{R}} \frac{\beta_{y_1-y}^{z_1+z}(s)}{|t-s|^{z+1}}\left(e^{-iy\pi}e^{-i(z+1)\frac{\pi}{2}\,\mathrm{sign}(t-s)} + e^{iy\pi}e^{i(z+1)\frac{\pi}{2}\,\mathrm{sign}(t-s)}\right)\mathrm{d}s \\
&= \frac{\Gamma(z+1)}{2\pi}\left(\kappa(z,y)\,{}_{-\infty}\mathbf{D}_t^z \beta_{y_1-y}^{z_1+z}(t) + \kappa(z,-y)\,{}_{-\infty}\mathbf{D}_{-t}^z \beta_{y_1-y}^{z_1+z}(-t)\right)
\end{aligned}$$

with $\kappa(z,y)$ and ${}_{-\infty}\mathbf{D}_t^z$ as given in the theorem. This concludes the proof. □

Note 4. Using integration by parts and Cauchy's principle value [24, p. 41 f.], ${}_{-\infty}\mathbf{D}_t^\alpha f(s)$, $f \in \mathcal{D}(\mathbb{R})$, can be reduced to a variant of the Sturm–Liouville or the Caputo fractional derivative. In fact, for $n-1 < \alpha < n$,

$$
\begin{aligned}
{}_{-\infty}\mathbf{D}_t^{\alpha} f(s) &= \int_{-\infty}^{t} \frac{f(s)}{(t-s)^{\alpha+1}}\,\mathrm{d}s \\
&= \lim_{\varepsilon\to 0} \frac{-f(t-\varepsilon)}{\varepsilon^{\alpha}\cdot\alpha} - \frac{f'(t-\varepsilon)}{\varepsilon^{\alpha-1}\cdot\alpha(\alpha-1)} - \cdots - \frac{f^{(n-1)}(t-\varepsilon)}{\varepsilon^{\alpha+1-n}\cdot\alpha(\alpha+1)\cdots(\alpha+1-n)} \\
&\quad + \int_{-\infty}^{t} \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}\cdot\alpha(\alpha+1)\cdots(\alpha+1-n)}\,\mathrm{d}s.
\end{aligned}
$$

The last term is a scaled version of Caputo fractional derivative (see [25])

$$
{}_a^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha+n)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}\,\mathrm{d}s \quad (0 \leqslant n-1 < \alpha < n), \tag{15}
$$

which for $\alpha = -\infty$ is equivalent to the Sturm–Liouville fractional derivative. These operators provide an interpolation between integer–order derivatives. Indeed, if a function f has $n+1$ continuous and bounded derivatives in $]-\infty, t]$, which vanish at $-\infty$, then $\lim_{\alpha\to n} {}_{-\infty}^{C} D_t^{\alpha} f(t) = f^{(n)}(t)$.

The real-valued version of the Caputo fractional differential operator is used for the analysis of steady state processes; e.g., fractional order dynamic systems with periodic input signals, wave propagation in viscoelastic materials, etc. [25].

6. Multiresolution analyses with complex B-splines

As their real-valued cousins, the complex B-splines generate dyadic multiresolution analyses; i.e., they generate a sequence of spaces

$$
\{0\} \subset \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2(\mathbb{R})
$$

with the following properties:

(i) $\bigcap_j V_j = \{0\}$ and $\overline{\bigcup_j V_j} = L^2(\mathbb{R})$,
(ii) $f \in V_j$ if and only if $f(2^{-j}\bullet) \in V_0$,
(iii) $f \in V_0$ if and only if $f(\bullet - k) \in V_0$ for all $m \in \mathbb{Z}$, and
(iv) there exists a function $\varphi \in V_0$, called a scaling function, such that $\{\varphi(\bullet - k)\}_{k\in\mathbb{Z}}$ forms an orthonormal basis of V_0.

The key property of a multiresolution analysis is a two-scale relation. It relates the spaces V_j and V_{j+1} by a dilation of their generators (here, the complex B-splines).

Proposition 8. *The complex B-splines β^z and β_y^z satisfy the following two-scale relations*:

$$
\beta^z(x) = 2^{-z} \sum_{k\geqslant 0} \binom{z+1}{k} \beta^z(2x-k)
$$

and

$$
\beta_y^z(x) = 2^{-z} \sum_{k\in\mathbb{Z}} \left| \begin{matrix} z+1 \\ y+k \end{matrix} \right| \beta_y^z(2x-k),
$$

for $\operatorname{Re} z > 0$ *and* $y \in \mathbb{C}$ *and almost all* $x \in \mathbb{R}$.

Proof. If there exists a two-scale relation $\beta^z = \sum_{k\in\mathbb{Z}} h_k \beta^z(2\bullet - k)$, then the $(h_k)_{k\in\mathbb{Z}}$ are the Fourier coefficients of the following frequency response of the refinement filter:

$$H^z(\omega) = 2\frac{\hat{\beta}^z(2\omega)}{\hat{\beta}^z(\omega)} = 2\cdot\frac{(i\omega)^{z+1}}{(2i\omega)^{z+1}}\frac{(1-e^{-2i\omega})^{z+1}}{(1-e^{-i\omega})^{z+1}} = \frac{2}{2^{z+1}}\left(1+e^{-i\omega}\right)^{z+1} = \frac{1}{2^z}\sum_{k\geqslant 0}\binom{z+1}{k}e^{-i\omega k} \tag{16}$$

almost everywhere. This function is clearly 2π-periodic, and thus corresponds to a digital convolution operator, the refinement filter.

For β_y^z we get

$$\begin{aligned} H_y^z(\omega) &= 2\frac{\hat{\beta}_y^z(2\omega)}{\hat{\beta}_y^z(\omega)} = 2\frac{\left(\frac{1-e^{-i2\omega}}{i2\omega}\right)^{\frac{z+1}{2}-y}}{\left(\frac{1-e^{-i\omega}}{i\omega}\right)^{\frac{z+1}{2}-y}}\frac{\left(\frac{1-e^{i2\omega}}{-i2\omega}\right)^{\frac{z+1}{2}+y}}{\left(\frac{1-e^{i\omega}}{-i\omega}\right)^{\frac{z+1}{2}+y}} = 2^{-z}\left(1+e^{-i\omega}\right)^{\frac{z+1}{2}-y}\left(1+e^{i\omega}\right)^{\frac{z+1}{2}+y} \\ &= H_y^z(\omega+2\pi) \quad \text{almost everywhere.} \end{aligned} \tag{17}$$

To deduce the Fourier coefficients of H_y^z, we use the Cauchy product formula with the generalized Vandermonde–Chu convolution formula and derive as in (5)

$$\begin{aligned} H_y^z(\omega) &= 2^{-z}\sum_{k\geqslant 0}\binom{\frac{z+1}{2}-y}{k}e^{-ik\omega}\sum_{l\geqslant 0}\binom{\frac{z+1}{2}+y}{l}e^{il\omega} \\ &= 2^{-z}\sum_{n\in\mathbb{Z}}\binom{z+1}{\frac{z+1}{2}-y-n}e^{-i\omega n} = 2^{-z}\sum_{n\in\mathbb{Z}}\left|\binom{z+1}{y+n}\right|e^{-i\omega n}. \end{aligned}$$

This concludes the proof. □

Theorem 9. *Let* $\operatorname{Re} z > 0$. *Then the spaces*

$$V_j = \overline{\operatorname{span}\left\{\beta^z\left(\frac{x-2^jk}{2^j}\right)\colon k\in\mathbb{Z}\right\}}^{L^2(\mathbb{R})}, \quad j\in\mathbb{Z},$$

resp.

$$V_{j,*} = \overline{\operatorname{span}\left\{\beta_y^z\left(\frac{x-2^jk}{2^j}\right)\colon k\in\mathbb{Z}\right\}}^{L^2(\mathbb{R})}, \quad j\in\mathbb{Z},$$

form dyadic multiresolution analyses with scaling function β^z, *resp.* β_y^z.

Proof. To prove that β^z generates a multiresolution analysis, we have to check the following three conditions [26, Theorem 2.13]: (i) $\{\beta^z(\bullet-k)\}_{k\in\mathbb{Z}}$ is a Riesz sequence in $L^2(\mathbb{R})$, (ii) the existence of two-scale relation, which was already shown in Proposition 8, and (iii) that $\hat{\beta}^z$ is continuous at the origin and $\hat{\beta}^z(0)=0$. From (i) we can deduce that $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$; (i) and (iii) give the density of $\bigcup_{j\in\mathbb{Z}} V_j$ in $L^2(\mathbb{R})$.

We first show that $\{\beta^z(x-k)\}_{k\in\mathbb{Z}}$ forms a Riesz basis of V_0. It is enough to show the existence of constants A and B such that

$$0 < A \leqslant \sum_{k\in\mathbb{Z}}\left|\hat{\beta}^z(\omega-2\pi k)\right|^2 \leqslant B < 0 \quad \text{almost everywhere.}$$

The central part can be rewritten as:

$$\sum_{k\in\mathbb{Z}}\left|\hat{\beta}^z(\omega+2\pi k)\right|^2 = \sum_{k\in\mathbb{Z}}\left|\hat{\beta}_+^\alpha(\omega+2\pi k)\right|^2\left|e^{-i\gamma\ln|\Omega(\omega+2\pi k)|}\right|^2\left|e^{\gamma\arg\Omega(\omega+2\pi k)}\right|^2. \tag{18}$$

For the fractional B-splines β_+^α, it is known [9, Proposition 3.3] that there are positive constants A, B such that $A \leqslant \sum_{k\in\mathbb{Z}}|\hat{\beta}_+^\alpha(\omega+2\pi k)|^2 \leqslant B$ for almost all $\omega\in\mathbb{R}$. Since $e^{-2\pi|\gamma|} \leqslant |e^{\gamma\arg\Omega(\omega+2\pi k)}|^2 \leqslant e^{2\pi|\gamma|}$, we deduce that $\{\beta^z(\bullet-k)\}_{k\in\mathbb{Z}}$ is a Riesz basis in V_0. Moreover, $\hat{\varphi}^z = \hat{\beta}^z/\sqrt{\sum_{k\in\mathbb{Z}}|\hat{\beta}^z(\bullet+2\pi k)|^2}$ generates an orthonormal basis of V_0.

The remaining step is to prove (iii), which is obvious from the definition of the complex B-spline. This concludes the proof for β^z. An analog argumentation is applicable as well for β_y^z and $V_{0,*}$. □

Using the fact that $\beta^z \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we can deduce from (9) that $\beta^z(2\pi k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$. The Poisson summation formula yields the partition of unity: $\sum_{k\in\mathbb{Z}} \beta^z(x-k) = 1$ almost everywhere (see also [27]).

To avoid exploding growth of the Riesz bounds in (18) it is advisable to keep γ reasonably small; say, $\gamma \in [-1, 1]$.

Note 5. Since the filter H_*^z has infinitely many non-vanishing Fourier coefficients, an implementation of the corresponding multiresolution algorithm via filtering as given in [15] is inpractical. As we saw in (17), the filter can be represented in an easily accessible closed form. This suggests an efficient implementation of the filtering algorithm in Fourier domain, as proposed in [18].

7. Gaussian shape

The classical B-splines β^n are known to converge to Gaussians [28]. In the following we examine to which extend the same is true for the complex B-splines.

Theorem 10. *Let $z = \alpha + i\gamma$ and $y \in \mathbb{C}$. Then, for $\alpha \to \infty$, and fixed $\gamma \in \mathbb{R}$ the complex B-splines converge pointwise in the Fourier domain to modulated and shifted Gaussians*:

$$\lim_{\alpha\to\infty} \hat{\beta}^z\left(\frac{\omega}{\sqrt{\alpha+1}}\right) \Big/ \left(e^{-\frac{i}{2}\sqrt{\alpha+1}\omega} e^{-\left(\frac{\omega}{2\sqrt{6}} - \frac{\gamma\sqrt{3}}{\sqrt{2}\sqrt{\alpha+1}}\right)^2} e^{\frac{3\gamma^2}{2(\alpha+1)}}\right) = \lim_{\alpha\to\infty} 1 + \mathcal{O}\left(\frac{\omega^2}{\alpha+1}\right) = 1, \tag{19}$$

resp.

$$\lim_{\alpha\to\infty} \hat{\beta}_y^z\left(\frac{\omega}{\sqrt{\alpha+1}}\right) \Big/ \left(e^{iy\frac{\omega}{\sqrt{\alpha+1}}} e^{-\frac{\omega^2}{24}}\right) = \lim_{\alpha\to\infty} 1 + \mathcal{O}\left(\frac{\gamma\omega^2}{\alpha+1}\right) = 1. \tag{20}$$

From these estimates we see that the parameter γ (resp., y) act like a frequency shift in $\hat{\beta}^z$ (resp., $\hat{\beta}_y^z$). This convergence process is illustrated in Fig. 7.

Proof. As $\omega \to 0$, we have that

$$\frac{1-e^{-i\omega}}{i\omega} = e^{-i\frac{\omega}{2}}\left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right) = e^{-i\frac{\omega}{2}} \cdot \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!}\left(\frac{\omega}{2}\right)^{2k} = e^{-i\frac{\omega}{2}}\left(1 - \frac{\omega^2}{24} + \frac{\omega^4}{2^4\cdot 4!} + \mathcal{O}(\omega^6)\right). \tag{21}$$

Thus

$$\ln\left(\frac{1-e^{-i\omega}}{i\omega}\right) = \ln\left(e^{-i\frac{\omega}{2}}\left(1 - \frac{\omega^2}{24} + \frac{\omega^4}{2^4\cdot 4!} + \mathcal{O}(\omega^6)\right)\right) = -i\frac{\omega}{2} + \ln\left(1 - \frac{\omega^2}{24} + \frac{\omega^4}{384} + \mathcal{O}(\omega^6)\right). \tag{22}$$

Now, we make use of the following expansion valid in a neighborhood of $x = 1$:

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \mathcal{O}\big((x-1)^4\big).$$

Thus, for $\omega \to 0$, we have

$$\ln\left(1 - \frac{\omega^2}{24} + \frac{\omega^4}{384} + \mathcal{O}(\omega^6)\right) = -\frac{\omega^2}{24} + \frac{1}{564}\omega^4 + \mathcal{O}(\omega^6),$$

and together with (22)

$$\ln\left(\frac{1-e^{-i\omega}}{i\omega}\right) = -\frac{i}{2}\omega - \frac{\omega^2}{24} + \frac{1}{564}\omega^4 + \mathcal{O}(\omega^6).$$

This yields

$$\hat{\beta}^z\left(\frac{\omega}{\sqrt{\alpha+1}}\right) = e^{-\frac{i}{2}\sqrt{\alpha+1}\omega} e^{-\left(\frac{\omega}{2\sqrt{6}} - \frac{\gamma\sqrt{3}}{\sqrt{\alpha+1}\sqrt{2}}\right)^2 + \frac{\gamma^2 3}{2(\alpha+1)} + \mathcal{O}\left(\frac{\omega^2}{\alpha+1}\right)},$$

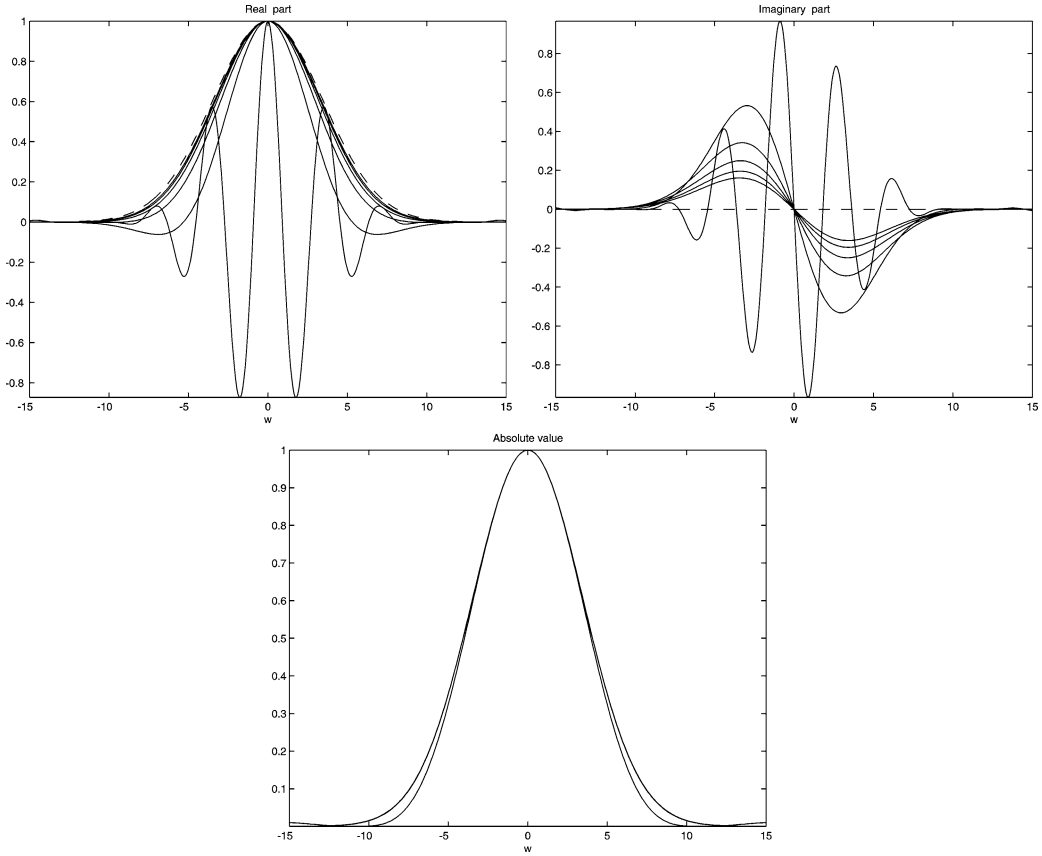


Fig. 7. Convergence of $\hat{\beta}^{\alpha}_{3i}(\omega)$ to a Gaussian (dotted line) $e^{-\omega^2/24}$ for $\alpha = 2, 3, \ldots, 6$. Already for small α, the real part and absolute value tend to a Gaussian, whereas the imaginary part tends to zero.

resp.

$$\hat{\beta}^z_y\left(\frac{\omega}{\sqrt{\alpha+1}}\right) = e^{iy\frac{\omega}{\sqrt{\alpha+1}}} e^{-\frac{\omega^2}{24}+\mathcal{O}(\frac{\omega^4}{\alpha+1})} e^{\frac{i\gamma}{\alpha+1}(-\frac{\omega^2}{24}+\mathcal{O}(\frac{\omega^4}{\alpha+1}))}$$

for $\alpha \to \infty$, $y \in \mathbb{C}$, pointwise for all $\omega \in \mathbb{R}$. Thus

$$\hat{\beta}^z\left(\frac{\omega}{\sqrt{\alpha+1}}\right) e^{\frac{i}{2}\sqrt{\alpha+1}\omega} e^{(\frac{\omega}{2\sqrt{6}}-\frac{\gamma\sqrt{3}}{\sqrt{\alpha+1}\sqrt{2}})^2-\frac{\gamma^2 3}{2(\alpha+1)}} = 1 + \mathcal{O}\left(\frac{\omega^2}{\alpha+1}\right),$$

resp.

$$\hat{\beta}^z_y\left(\frac{\omega}{\sqrt{\alpha+1}}\right) e^{-iy\frac{\omega}{\sqrt{\alpha+1}}+\frac{\omega^2}{24}} = 1 + \mathcal{O}\left(\frac{i\gamma\omega^2}{\alpha+1}\right)$$

for $\alpha \to \infty$. This gives assertions (19) and (20). □

Note 6. From the above calculations one deduces that

$$\hat{\beta}^z_y\left(\frac{\omega}{\sqrt{\alpha+1}}\right) = e^{-\omega^2/24}\left(1 + \frac{iy\omega}{\sqrt{\alpha+1}} + i\gamma\frac{\omega^2}{24(\alpha+1)} + \mathcal{O}\left(\frac{\omega^4}{\alpha+1}\right)\right)$$

pointwise for $\alpha \to \infty$ and fixed $y \in \mathbb{C}$. Thus, B-splines of fractional degree converge in order $\mathcal{O}(1/(\alpha+1))$, whereas the shifted splines converge in $\mathcal{O}(1/\sqrt{\alpha+1})$.

Theorem 11. *The complex B-splines converge to modulated Gaussians in $L^p(\mathbb{R})$ for $1 \leqslant p \leqslant \infty$:*

$$\left\| \hat{\beta}^z\left(\frac{\bullet}{\sqrt{\alpha+1}}\right) - e^{-\frac{i}{2}\sqrt{\alpha+1}\omega} e^{-(\frac{\omega}{2\sqrt{6}} - \frac{\gamma\sqrt{3}}{\sqrt{\alpha+1}\sqrt{2}})^2 + \frac{\gamma^2 3}{2(\alpha+1)}} \right\|_p \to 0$$

and

$$\left\| \hat{\beta}^z_y\left(\frac{\bullet}{\sqrt{\alpha+1}}\right) - e^{iy\frac{\omega}{\sqrt{\alpha+1}} - \frac{\omega^2}{24}} \right\|_p \to 0$$

for $\alpha \to \infty$.

In time domain for $2 \leqslant q \leqslant \infty$

$$\left\| \sqrt{\alpha+1}\,\beta^z(\sqrt{\alpha+1}\,\bullet) - \sqrt{\frac{6}{\pi}} e^{\frac{3\gamma^2}{2(\alpha+1)}} e^{i\frac{6\gamma}{\sqrt{\alpha+1}}(\bullet - \frac{\sqrt{\alpha+1}}{2})} e^{-(\frac{1}{2}\sqrt{6}(\bullet - \frac{\sqrt{\alpha+1}}{2}))^2} \right\|_q \to 0,$$

resp.

$$\left\| \sqrt{\alpha+1}\,\beta^z_y(\sqrt{\alpha+1}\,\bullet) - \sqrt{\frac{6}{\pi}} e^{-\frac{6y^2}{\alpha+1}} e^{-12\frac{y}{\sqrt{\alpha+1}}\bullet} e^{-6(\bullet)^2} \right\|_q \to 0$$

for $\alpha \to \infty$.

The rate of convergence in all four cases is $\mathcal{O}(1/(\alpha+1))$.

The limits in time domain are also true pointwise.

Proof. In [28] it is shown that

$$\left(\frac{\sin\frac{\pi\omega}{\sqrt{\alpha+1}}}{\frac{\pi\omega}{\sqrt{\alpha+1}}}\right)^{\alpha+1} \leqslant e^{-\omega^2} + (1 - \chi_{[-1,1]})\left(\frac{\omega}{2}\right) \cdot \frac{2}{(\pi\omega)^2}$$

for all $\omega \in \mathbb{R}$ and all $\alpha \geqslant 1$. The function on the right-hand side is integrable, in $L^p(\mathbb{R})$ for $1 \leqslant p \leqslant \infty$, and independent of α.

Thus we have

$$\begin{aligned} \left|\hat{\beta}^z\left(\frac{\omega}{\sqrt{\alpha+1}}\right)\right| &\leqslant \left|\hat{\beta}^\alpha\left(\frac{\omega}{\sqrt{\alpha+1}}\right)\right| \cdot e^{2\pi|\gamma|} \leqslant \left(\frac{\sin\frac{\omega}{2\sqrt{\alpha+1}}}{\frac{\omega}{2\sqrt{\alpha+1}}}\right)^{\alpha+1} \cdot e^{2\pi|\gamma|} \\ &\leqslant \left(e^{-\frac{\omega^2}{(4\pi^2)}} + (1 - \chi_{[-1,1]})\left(\frac{\omega}{4\pi}\right) \cdot \frac{2}{(\omega/2)^2}\right) \cdot e^{2\pi|\gamma|}, \end{aligned} \tag{23}$$

and an analog result is true for $\hat{\beta}^z_y$ and $\hat{\beta}^\alpha_*$. For the approximant we have

$$\left|e^{(\alpha+1+i\gamma)(-\frac{i}{2}\frac{\omega}{\sqrt{\alpha+1}} - \frac{\omega^2}{24(\alpha+1)})}\right| = \left|e^{\frac{\gamma}{2}\frac{\omega}{\sqrt{\alpha+1}} e^{-\frac{\omega^2}{24}}}\right| = \left|e^{-\frac{1}{24}(\omega - 12\frac{\gamma}{\sqrt{\alpha+1}})^2 + 6\frac{\gamma^2}{\alpha+1}}\right| \leqslant c e^{6\gamma^2} e^{-\frac{1}{24}\omega^2}. \tag{24}$$

The majorants both in (23) and (24) are independent of α and, thus, by Lebesgue's dominated convergence theorem, we have convergence in $L^p(\mathbb{R})$, $1 \leqslant p < \infty$:

$$\begin{aligned} &\left\| \hat{\beta}^z\left(\frac{\bullet}{\sqrt{\alpha+1}}\right) - e^{-\frac{i}{2}\sqrt{\alpha+1}\bullet} e^{-(\frac{\bullet}{2\sqrt{6}} - \frac{\gamma\sqrt{3}}{\sqrt{\alpha+1}\sqrt{2}})^2 + \frac{\gamma^2 3}{2(\alpha+1)}} \right\|_p \\ &\quad = \left\| e^{-(\frac{\bullet}{2\sqrt{6}} - \frac{\gamma\sqrt{3}}{\sqrt{\alpha+1}\sqrt{2}})^2 + \frac{\gamma^2 3}{2(\alpha+1)}} \left(e^{\mathcal{O}(\frac{(\bullet)^2}{\alpha+1})} - 1\right) \right\|_p = \mathcal{O}\left(\frac{1}{\alpha+1}\right) \to 0 \end{aligned} \tag{25}$$

and the same for

$$\left\| \hat{\beta}^z_y\left(\frac{\bullet}{\sqrt{\alpha+1}}\right) - e^{iy\frac{\bullet}{\sqrt{\alpha+1}} - \frac{(\bullet)^2}{24}} \right\|_p = \mathcal{O}\left(\frac{1}{\alpha+1}\right) \to 0$$

for $\alpha \to \infty$. Since both $\hat{\beta}^z, \hat{\beta}^z_y \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ we find by the Hausdorff–Young inequality [29] and Fourier inversion

$$\left\| \sqrt{\alpha+1}\beta^z(\sqrt{\alpha+1}\,\bullet) - \sqrt{\frac{6}{\pi}} e^{\frac{3\gamma^2}{2(\alpha+1)}} e^{i\frac{6\gamma}{\sqrt{\alpha+1}}(\bullet-\frac{\sqrt{\alpha+1}}{2})} e^{-(\frac{1}{2}\sqrt{6}(\bullet-\frac{\sqrt{\alpha+1}}{2})^2)} \right\|_q$$
$$\leqslant C \left\| \hat{\beta}^z\left(\frac{\bullet}{\sqrt{\alpha+1}}\right) - e^{-\frac{i}{2}\sqrt{\alpha+1}\bullet} e^{-(\frac{\bullet}{2\sqrt{6}} - \frac{\gamma\sqrt{3}}{\sqrt{\alpha+1}\sqrt{2}})^2 + \frac{\gamma^2 3}{2(\alpha+1)}} \right\|_p = \mathcal{O}\left(\frac{1}{\alpha+1}\right) \to 0$$

for $\alpha \to \infty$ and

$$\left\| \sqrt{\alpha+1}\,\beta_y^z(\sqrt{\alpha+1}\,\bullet) - \sqrt{\frac{6}{\pi}} e^{-\frac{6y^2}{\alpha+1}} e^{-12\frac{y}{\sqrt{\alpha+1}}\bullet} e^{-6(\bullet)^2} \right\|_q$$
$$\leqslant C \left\| \hat{\beta}_y^z\left(\frac{\bullet}{\sqrt{\alpha+1}}\right) - e^{-iy\frac{\bullet}{\sqrt{\alpha+1}} + \frac{(\bullet)^2}{24}} \right\|_p = \mathcal{O}\left(\frac{1}{\alpha+1}\right) \to 0 \quad \text{for } \alpha \to \infty,$$

for $1 \leqslant p \leqslant 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. In both cases, the constant C is positive and only depends on p.

Theorem 10 already gave the pointwise convergence in the Fourier domain. Since both $\hat{\beta}^z$ and $\hat{\beta}_y^z$ are elements of $L^1(\mathbb{R})$, we deduce the corresponding pointwise convergence in time domain by Fourier inversion and the dominated convergence theorem. □

8. The Heisenberg uncertainty bound for β_y^z

The asymptotics of the previous sections give hope that the complex B-splines β_y^z converge to optimally time–frequency localized functions in the sense of Heisenberg; i.e.,

$$\frac{1}{2\pi} \int_{\mathbb{R}} x^2 \left|\beta_y^z(x)\right|^2 \mathrm{d}x \cdot \int_{\mathbb{R}} \omega^2 \left|\hat{\beta}_y^z(\omega)\right|^2 \mathrm{d}\omega \geqslant \frac{1}{4} \|\beta_y^z\|_2^4,$$

with equality as $\operatorname{Re} z = \alpha \to \infty$.

Theorem 12. *For* $\operatorname{Re} z = \alpha \to \infty$, *the complex B-spline* β_y^z *satisfies the lower bound of the Heisenberg uncertainty principle*:

$$\frac{1}{2} \leqslant \frac{\|(\sqrt{\alpha+1}\bullet)\beta_y^z(\sqrt{\alpha+1}\bullet)\|_2}{\|\beta_y^z(\sqrt{\alpha+1}\bullet)\|_2} \cdot \frac{\left\|\left(\frac{\bullet}{\sqrt{\alpha+1}}\right)\hat{\beta}_y^z\left(\frac{\bullet}{\sqrt{\alpha+1}}\right)\right\|_2}{\left\|\hat{\beta}_y^z\left(\frac{\bullet}{\sqrt{\alpha+1}}\right)\right\|_2}$$
$$= \frac{1}{2} + \mathcal{O}\left(\frac{y}{\sqrt{\alpha+1}}\right) + \mathcal{O}\left(\frac{\gamma}{\alpha+1}\right) \to \frac{1}{2} \quad \textit{for } \alpha \to \infty. \tag{26}$$

Proof. We show that β_y^α is approximately optimally time–frequency-localized for $\alpha \to \infty$, i.e., its Heisenberg uncertainty product converges to $\frac{1}{2}$. Therefore, we use the asymptotics for β_y^z:

$$\lim_{\alpha\to\infty} \hat{\beta}_y^z\left(\frac{\omega}{\sqrt{\alpha+1}}\right) = e^{-\omega^2/24} =: \hat{G}(\omega),$$
$$\lim_{\alpha\to\infty} \sqrt{\alpha+1}\beta_y^z(\sqrt{\alpha+1}x) = \sqrt{\frac{6}{\pi}} e^{-6x^2} = G(x). \tag{27}$$

Applying the triangle inequality for α large enough, we get

$$\frac{1}{2} \leqslant \frac{\|(\bullet)\beta_y^z(\sqrt{\alpha+1}\bullet)\|_2}{\|\beta_y^z(\sqrt{\alpha+1}\bullet)\|_2} \cdot \frac{\left\|(\bullet)\hat{\beta}_y^z\left(\frac{\bullet}{\sqrt{\alpha+1}}\right)\right\|_2}{\left\|\hat{\beta}_y^z\left(\frac{\bullet}{\sqrt{\alpha+1}}\right)\right\|_2}$$
$$= \frac{\|(\bullet)(\beta_y^z(\sqrt{\alpha+1}\bullet) - G(\bullet) + G(\bullet))\|_2}{\|\beta_y^z(\sqrt{\alpha+1}\bullet) - G(\bullet) + G(\bullet)\|_2} \cdot \frac{\left\|(\bullet)\left(\hat{\beta}_y^z\left(\frac{\bullet}{\sqrt{\alpha+1}}\right) - \hat{G} + \hat{G}\right)\right\|_2}{\left\|\hat{\beta}_y^z\left(\frac{\bullet}{\sqrt{\alpha+1}}\right) - \hat{G} + \hat{G}\right\|_2}$$

$$\leqslant \frac{\|(\bullet)(\beta_y^z(\sqrt{\alpha+1}\bullet)-G)\|_2+\|(\bullet)G\|_2}{\|G\|_2-\|\beta_y^z(\sqrt{\alpha+1}\bullet)-G\|_2}\,\frac{\|(\bullet)(\hat\beta_y^z(\frac{\bullet}{\sqrt{\alpha+1}})-\hat G)\|_2+\|(\bullet)\hat G\|_2}{\|\hat G\|_2-\|\hat\beta_y^z(\frac{\bullet}{\sqrt{\alpha+1}})-\hat G\|_2}.$$

Since we already know from the proof of Theorems 10 and 11 that

$$\lim_{\alpha\to\infty}\|\beta_y^z(\sqrt{\alpha+1}\bullet)-G\|_2=\lim_{\alpha\to\infty}\left\|\hat\beta_y^z\left(\frac{\bullet}{\sqrt{\alpha+1}}\right)-\hat G\right\|_2=\lim_{\alpha\to\infty}\mathcal{O}\left(\frac{y}{\sqrt{\alpha+1}}\right)+\mathcal{O}\left(\frac{\gamma}{\alpha+1}\right)=0,$$

and

$$\frac{\|(\bullet)G\|_2}{\|G\|_2}\cdot\frac{\|(\bullet)\hat G\|_2}{\|\hat G\|_2}=\frac{1}{2},$$

it is sufficient to show

$$\lim_{\alpha\to\infty}\|(\bullet)(\beta_y^z(\sqrt{\alpha+1}\bullet)-G)\|_2=0 \tag{28}$$

and

$$\lim_{\alpha\to\infty}\left\|(\bullet)\left(\hat\beta_y^z\left(\frac{\bullet}{\sqrt{\alpha+1}}\right)-\hat G\right)\right\|_2=0. \tag{29}$$

In fact, (29) can be deduced from (23) and (24) with Lebesgue's theorem of dominated convergence, since the multiplication with ω leaves the terms independent of α and does not brake $L^2(\mathbb{R})$ integrability for α large enough. Moreover, (29) converges at the rate $\mathcal{O}(y/\sqrt{\alpha+1})+\mathcal{O}(\gamma/(\alpha+1))$.

To verify (28), we note that

$$\hat G'(\omega)=-\frac{\omega}{12}e^{-\omega^2/24},$$

and that it is equivalent to show that

$$\lim_{\alpha\to\infty}\left\|(\hat\beta_y^z)'\left(\frac{\bullet}{\sqrt{\alpha+1}}\right)\frac{1}{\sqrt{\alpha+1}}-\hat G'\right\|_2=0, \tag{30}$$

since multiplication with polynomials corresponds to differentiation in Fourier domain. We first prove pointwise convergence, and then find a majorant independent of α.

For $(\hat\beta_y^z)'$, we have

$$\begin{aligned}(\hat\beta_y^z)'(\omega)&=\hat\beta_y^{z-2}(\omega)\frac{4\sin(\frac{\omega}{2})(\frac{z+1}{2}\omega\cos(\frac{\omega}{2})-(z+1)\sin(\frac{\omega}{2})+iy\omega\sin(\frac{\omega}{2}))}{\omega^3}\\&=\hat\beta_y^{z-2}(\omega)\left(-\frac{z+1}{12}\omega+(z+1)\mathcal{O}(\omega^3)+iy-iy\mathcal{O}(\omega^2)\right)\quad\text{for }\omega\to 0.\end{aligned} \tag{31}$$

Now, we consider $\frac{1}{\sqrt{\alpha+1}}(\hat\beta_y^z)'(\frac{\omega}{\sqrt{\alpha+1}})$ as needed in (30).

$$\begin{aligned}\frac{1}{\sqrt{\alpha+1}}(\hat\beta_y^z)'\left(\frac{\omega}{\sqrt{\alpha+1}}\right)&=\frac{1}{\sqrt{\alpha+1}}\hat\beta_y^{z-2}\left(\frac{\omega}{\sqrt{\alpha+1}}\right)\\&\quad\times\left(-\frac{z+1}{12}\frac{\omega}{\sqrt{\alpha+1}}+\frac{z+1}{\sqrt{\alpha+1}^3}\mathcal{O}(\omega^3)+iy-\frac{iy}{\alpha+1}\mathcal{O}(\omega^2)\right)\\&=\hat\beta_y^{z-2}\left(\frac{\omega}{\sqrt{\alpha+1}}\right)\left(-\frac{\omega}{12}-\frac{i\gamma\omega}{12(\alpha+1)}+\mathcal{O}\left(\frac{\omega^3}{\alpha+1}\right)+\frac{iy}{\sqrt{\alpha+1}}-iy\mathcal{O}\left(\frac{\omega^2}{\sqrt{\alpha+1}^3}\right)\right)\\&=-\frac{\omega}{12}e^{-\frac{\omega^2}{24}}+\mathcal{O}\left(\frac{y}{\sqrt{\alpha+1}}\right)+\mathcal{O}\left(\frac{\gamma}{\alpha+1}\right)\quad\text{for }\alpha\to\infty\end{aligned}$$

and all fixed $\omega\in\mathbb{R}$. Here, we used (27) for the convergence of $\hat\beta_y^{z-2}$.

We show that the function (31) has an integrable majorant, which is independent of α. In the first step, we consider the influence of y. Then we estimate the remaining terms.

Step 1. Clearly,

$$\left|\left(\hat{\beta}_y^z\right)'(\omega)\right| \leqslant \left|\hat{\beta}_y^{z-2}(\omega)\right| \cdot \left|\tilde{f}(\omega)\right| \leqslant \left|\hat{\beta}_*^{\alpha-2}(\omega)\right| \cdot \left|\tilde{f}(\omega)\right| \cdot e^{2\pi|\eta|},$$

where

$$\tilde{f}(\omega) = \frac{4\sin\left(\frac{\omega}{2}\right)\left(\frac{z+1}{2}\omega\cos\left(\frac{\omega}{2}\right) - (z+1)\sin\left(\frac{\omega}{2}\right) + iy\omega\sin\left(\frac{\omega}{2}\right)\right)}{\omega^3} = f(\omega) + 4iy\frac{\sin^2\left(\frac{\omega}{2}\right)}{\omega^2}. \tag{32}$$

Thus,

$$\left|\frac{1}{\sqrt{\alpha+1}}\left(\hat{\beta}_y^z\right)'\left(\frac{\omega}{\sqrt{\alpha+1}}\right)\right| \leqslant \left|\hat{\beta}_*^{\alpha-2}\left(\frac{\omega}{\sqrt{\alpha+1}}\right)\right| \cdot \left|\frac{1}{\sqrt{\alpha+1}}\tilde{f}\left(\frac{\omega}{\sqrt{\alpha+1}}\right)\right| \cdot e^{2\pi|\eta|}.$$

We estimate the y-dependent part in (32):

$$\left|\frac{4y}{\sqrt{\alpha+1}}\frac{\sin^2\left(\frac{\omega}{2\sqrt{\alpha+1}}\right)}{\left(\frac{\omega}{\sqrt{\alpha+1}}\right)^2}\right| = \left|\frac{y}{\sqrt{\alpha+1}}\frac{\sin^2\left(\frac{\omega}{2\sqrt{\alpha+1}}\right)}{\left(\frac{\omega}{2\sqrt{\alpha+1}}\right)^2}\right| \leqslant \frac{|y|}{\sqrt{\alpha+1}} \leqslant 1$$

for α large enough, and for all $\omega \in \mathbb{R}$. To estimate the function $\hat{\beta}_*^{\alpha-2}$, we follow [28].

$$\begin{aligned}\left|\hat{\beta}_*^{\alpha-2}\left(\frac{\omega}{\sqrt{\alpha+1}}\right)\right| &\leqslant \left|\frac{\sqrt{\alpha+1}}{\frac{\omega}{2}}\right|^{\alpha-1} = \left(\frac{\sqrt{\alpha-1}}{\frac{\omega}{2}}\right)^{\alpha-1} \cdot \left(\frac{\sqrt{\alpha+1}}{\sqrt{\alpha-1}}\right)^{\alpha-1} \\ &= \left|\frac{\sqrt{\alpha-1}}{\frac{\omega}{2}}\right|^{\alpha-1} \cdot \left(1 + \frac{2}{\alpha-1}\right)^{\frac{\alpha-1}{2}} \leqslant \frac{8e}{\omega^2}\end{aligned}$$

for all $\omega \geqslant \sqrt{\alpha+1}$. For $\omega \in [0, 2\sqrt{\alpha+1}]$, we make use of the fact that

$$\left|\hat{\beta}_*^{\alpha-2}(\omega)\right| = \left|\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right|^{\alpha-1} \leqslant \left(1 - \left(\frac{\omega}{2\pi}\right)^2\right)^{\alpha-1}$$

for all $\omega \in [0, 2\pi]$. Thus

$$\begin{aligned}\left|\hat{\beta}_*^{\alpha-2}\left(\frac{\omega}{\sqrt{\alpha+1}}\right)\right| &\leqslant \left(1 - \frac{\omega^2}{(2\pi)^2(\alpha+1)}\right)^{\alpha-1} \leqslant \left(1 - \frac{\omega^2}{4\pi^2(\alpha+1)}\right)^{\alpha+1} \frac{1}{\left(1 - \frac{\omega^2}{4\pi^2(\alpha+1)}\right)^2} \\ &\leqslant e^{-\frac{\omega^2}{4\pi^2}} \frac{1}{\left(1 - \frac{1}{\pi^2}\right)^2} \quad \text{for } \omega \in [0, 2\sqrt{\alpha+1}].\end{aligned} \tag{33}$$

Step 2. We first find an integrable majorant for the part

$$\hat{\beta}_*^{\alpha-2}\left(\frac{\omega}{\sqrt{\alpha+1}}\right) \cdot \frac{1}{\sqrt{\alpha+1}} f\left(\frac{\omega}{\sqrt{\alpha+1}}\right). \tag{34}$$

As we have seen before, $|\hat{\beta}_y^{z-2}(\omega)| \leqslant |\hat{\beta}_*^{\alpha-2}(\omega)| \cdot e^{2|\eta|\pi}$. Note that it is enough to consider $\omega > 0$, since $(\hat{\beta}_*^{\alpha})'(-\omega) = -(\hat{\beta}_*^{\alpha})'(\omega)$.

First case: $\omega \geqslant 2\sqrt{\alpha+1}$

We have $|f(\omega)| \leqslant 4|z+1|\frac{1}{\omega^3}(\frac{\omega}{2}+1)$. Thus

$$\begin{aligned}\frac{1}{\sqrt{\alpha+1}}\left|f\left(\frac{\omega}{\sqrt{\alpha+1}}\right)\right| &\leqslant 4|z+1|(\alpha+1)\frac{1}{\omega^3}\left(\frac{\omega}{2\sqrt{\alpha+1}} + 1\right) \leqslant 4|z+1|(\alpha+1)\left(\frac{1}{2\sqrt{\alpha+1}^3} + \frac{1}{\sqrt{\alpha+1}^3}\right) \\ &\leqslant 6\sqrt{\alpha+1+\gamma^2}.\end{aligned}$$

For the fractional splines, we get

$$\left|\hat{\beta}_*^{\alpha-2}\left(\frac{\omega}{\sqrt{\alpha+1}}\right)\right| \leqslant \left|\frac{2\sqrt{\alpha+1}}{\omega}\right|^{\alpha-1} = \left|\frac{\sqrt{\alpha+1}}{\frac{\omega}{2}}\right|^{\alpha-1}\left(\frac{\sqrt{\alpha+1}}{\sqrt{\alpha-1}}\right)^{\alpha-1} \leqslant \frac{3\sqrt{3}}{\left(\frac{\omega}{2}\right)^3}\cdot\left(1-\frac{2}{\alpha-1}\right)^{\frac{\alpha-1}{2}} \leqslant \frac{3\sqrt{3}\cdot e}{\left(\frac{\omega}{2}\right)^3}$$

for $\alpha \geqslant 4$ and $\omega \geqslant 2\sqrt{\alpha+1}$, since $\left(\frac{\sqrt{s}}{x}\right)^s \leqslant \frac{3\sqrt{3}}{x^3}$ for all $x > \sqrt{s}$ and $s \geqslant 3$. The complete term thus can be estimated with

$$\left|\hat{\beta}_y^{z-2}\left(\frac{\omega}{\sqrt{\alpha+1}}\right)\cdot\frac{1}{\sqrt{\alpha+1}}f\left(\frac{\omega}{\sqrt{\alpha+1}}\right)\right| \leqslant \frac{3\sqrt{3}e}{\left(\frac{\omega}{2}\right)^3}\cdot 6\sqrt{\alpha+1+\gamma^2}e^{2|\eta|\pi} \leqslant \frac{18\sqrt{3}e}{\left(\frac{\omega}{2}\right)^2}\sqrt{1+\gamma^2}e^{2|\eta|\pi},$$

for $\omega \geqslant 2\sqrt{\alpha+1}$ and $\alpha \geqslant 4$.

Second case: $\omega < 2\sqrt{\alpha+1}$

The factor $|\hat{\beta}_y^{z-2}(\omega)| \leqslant |\hat{\beta}_*^{\alpha-2}(\omega)|e^{2|\eta|\pi}$ in (34) is estimated as in (33).

For the second factor, we have

$$f(\omega) = \frac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}}\cdot\frac{z+1}{2}\cdot\frac{\frac{\omega}{2}\cos\left(\frac{\omega}{2}\right)-\sin\left(\frac{\omega}{2}\right)}{\left(\frac{\omega}{2}\right)^2}.$$

For $x \in [0,2]$, $\sin x - x\cos x \leqslant x^3$, since both sides vanish at zero and the left-hand side increases slower than the right-hand side in this interval. Thus for all $\omega \in [0, 2\sqrt{\alpha+1}]$

$$\left|\frac{1}{\sqrt{\alpha+1}}f\left(\frac{\omega}{\sqrt{\alpha+1}}\right)\right| \leqslant \frac{1}{\sqrt{\alpha+1}}\cdot\frac{|z+1|}{2}\cdot\frac{\omega}{\sqrt{\alpha+1}} \leqslant \frac{\omega}{2}\sqrt{1+\gamma^2}$$

for α large enough. Together, we find

$$\left|\frac{1}{\sqrt{\alpha+1}}f\left(\frac{\omega}{\sqrt{\alpha+1}}\right)\right| \leqslant \frac{1}{2\left(1-\frac{\sqrt{1+\gamma^2}}{\pi^2}\right)^2}e^{-\frac{\omega^2}{4\pi^2}}\omega \leqslant \frac{\sqrt{1+\gamma^2}}{2\left(1-\frac{1}{\pi^2}\right)^2}e^{-\frac{\omega^2}{8\pi^2}},$$

since $0 \leqslant e^{-\frac{\omega^2}{8\pi^2}}(1-\omega e^{-\frac{\omega^2}{8\pi^2}})$ is true for all $\omega > 0$.

Combining the first and second step, we find that for α large enough

$$\max\left(\frac{1}{2\left(1-\frac{1}{\pi^2}\right)^2}e^{-\frac{\omega^2}{8\pi^2}}, 18\sqrt{3}e\frac{1}{\left(\frac{\omega}{2}\right)^2}\left(1-\chi_{[-1,1]}(\omega)\right)\right)\cdot e^{2\pi|\eta|}\sqrt{1+\gamma^2}$$

is an integrable majorant for $\left|\frac{1}{\sqrt{\alpha+1}}(\hat{\beta}_*^z)'\left(\frac{\omega}{\sqrt{\alpha+1}}\right)\right|$ which is independent of α. This concludes the proof. □

Note 7. The order of convergence of the Heisenberg product to 1/2 is $\mathcal{O}(1/\sqrt{\alpha+1})$ for complex B-splines $\hat{\beta}_y^z$, and $\mathcal{O}(1/\alpha+1)$ for the fractional ones. Again, the shift in the frequency domain caused by the complex exponent results in a slower rate of convergence.

9. Conclusions

Complex B-splines appear to be a natural extension of the classical Schoenberg B-splines as well as fractional B-splines to a complex-valued setting. They satisfy all the properties of a scaling function and generate multiresolution analyses. The corresponding refinement filters are infinite impulse response filters, but due to the closed form of their frequency response, they are well suited for a Fourier domain implementation of the respective discrete wavelet transform. The complex B-splines are well localized in time as well as frequency domain, and converge to optimally time–frequency localized functions in the sense of Heisenberg. This and the four adjustable parameters, which allow to tune smoothness, modulation, and frequency enhancement, might be interesting for signal and image analysis applications.

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