An Unbiased Risk Estimator for Multiplicative Noise – Application to 1-D Signal Denoising

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***Abstract*—The effect of multiplicative noise on a signal when compared with that of additive noise is very large. In this paper, we address the problem of suppressing multiplicative noise in one-dimensional signals. To deal with signals that are corrupted with multiplicative noise, we propose a denoising algorithm based on minimization of an unbiased estimator (MURE) of mean-square error (MSE). We derive an expression for an unbiased estimate of the MSE. The proposed denoising is carried out in wavelet domain (soft thresholding) by considering time-domain MURE. The parameters of thresholding function are obtained by minimizing the unbiased estimator MURE. We show that the parameters for optimal MURE are very close to the optimal parameters considering the oracle MSE. Experiments show that the SNR improvement for the proposed denoising algorithm is competitive with a state-of-the-art method.**

***Index Terms*—multiplicative noise, denoising, risk estimation, thresholding.**

I. Introduction

A. Background

Signal denoising is a widely studied problem. Most of the literature deals with the additive noise model: given an original signal x, one typically assumes that it is corrupted by additive noise w. The problem is then to recover x from the measurements $y = x + w$. Many approaches have been proposed in the literature to suppress additive noise.

In this paper, we are concerned with a different denoising problem. The assumption is that the original signal x has been corrupted by multiplicative noise w: the goal is to estimate x from the measurement $y = xw$. Multiplicative noise is encountered in coherent imaging systems such as laser Doppler imaging, synthetic aperture radar (SAR) imaging, synthetic aperture sonar (SAS) imaging, ultrasound imaging, etc.

In general, the noise in the multiplicative noise model is described by a non-Gaussian probability density function with Gamma being the common employed model [1], [2].

B. Related Literature

Several methods have been proposed in the literature [3]–[13] to reduce the effect of multiplicative noise. One way is to transform the multiplicative noise in the signal to an additive one by employing a $\log$ transformation. Thereafter, additive suppression methods can be employed for restoring the signal. Finally, taking the exponential of the processed signal, we obtain the restored signal. However, such a straightforward method does not lead to satisfactory results. Most widely used filtering approaches to suppress multiplicative noise include Kuan filter [3], Lee filter [4], Frost filter [5] and adaptive speckle filtering by Lopes et al. [6]. Variational approaches using total variation (TV) were proposed by Rudin et al. [7], Aubert and Aujol [8], Shi and Osher [9], and Huang et al. [10]. Durand et al. proposed an approach by combining curvelet thresholding and variational method [11]. Bioucas-Dias and Figueiredo proposed a method based on variable splitting and constrained optimization [13].

C. This Paper

We propose a denoising method based on minimization of unbiased risk estimator. Since the oracle MSE (or risk) is unknown, we derive an expression for the unbiased estimator of MSE, namely multiplicative noise unbiased risk estimator (MURE). We find the optimal parameters of the denoising function by minimizing MURE, which yields a good estimate of the original signal. Unbiased risk estimation approaches have been developed for additive noise [14]–[16]. To the best of our knowledge, risk estimators for multiplicative noise have not been reported in the literature. We compare the performance of the proposed method with a state-of-the-art method.

II. Problem Statement

Consider the multiplicative noise signal model,

$$y_i = x_i w_i, \text{ for } i = 1, 2, \cdots, N,$$

where y_i is the i^{th} element of $\mathbf{y}$, $\mathbf{y} \in \mathcal{R}^N$ is the noise-contaminated observation of original (unknown) signal $\mathbf{x} \in \mathcal{R}^N$, $\{w_i,\ i = 1, 2, \cdots, N\}$ are identical and independent noise random variables with probability density function denoted by $q(w_i)$.

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The multiplicative noise is Gamma distributed with probability density function given as

$$q(\mathbf{w}) = \prod_{i=1}^{N} q(w_i), \text{ where } \mathbf{w} = [w_1, w_2, \cdots, w_N] \in \mathcal{R}^N,$$

$$q(w_i) = \frac{a^b}{\Gamma(b)} w_i^{b-1} \exp^{-w_i a}, \tag{1}$$

where a and $\frac{1}{b}$ are the scale and shape parameters, respectively.

$q(w_i)$ has mean $\mathcal{E}[q(w_i)] = a/b$, and variance

$$\sigma_w^2 = \mathcal{E}\left\{ [q(w_i) - \mathcal{E}(q(w_i))]^2 \right\} = \frac{a}{b^2}.$$

We choose $a = b = k$. Let $\mathbf{f} : \mathcal{R}^N \to \mathcal{R}^N$ be the denoising function that yields denoised estimate of original signal, $\hat{\mathbf{x}} = \mathbf{f}(\mathbf{y})$. We have taken a parametric form for $\mathbf{f}$. The goal is to find the optimal parameters of denoising function $\mathbf{f}$ such that

$$\text{MSE} = \frac{1}{N}\mathcal{E}\left\{ \|\hat{\mathbf{x}} - \mathbf{x}\|^2 \right\} \text{ is minimized.}$$

The ground truth signal is not known. Hence, instead of minimizing the oracle MSE, one needs to obtain an unbiased estimator of the MSE and minimize it.

III. Multiplicative Noise Unbiased Risk Estimator (MURE)

Notation. *Given a 1-D function f, we define $\mathcal{M}$ by the following operator*

$$\mathcal{M}f(y) = k\int_0^1 f(sy)s^{k-1}\,\mathrm{d}s.$$

For a multivariate function $\mathbf{f}(\mathbf{y}) = \mathbf{f}(y_1, y_2, \ldots, y_N)$, we define the operator $\mathcal{M}_i$ which applies the operator $\mathcal{M}$ to the i^{th} component of $\mathbf{f}(\mathbf{y})$ only.

This notation is extended straightforwardly to multivariate vector functions $\mathbf{f}(\mathbf{y}) = [f_1(\mathbf{y}), f_2(\mathbf{y}), \ldots, f_N(\mathbf{y})]^{\mathrm{T}}$ according to

$$\mathcal{M}\mathbf{f}(\mathbf{y}) = [\mathcal{M}_1 f_1(\mathbf{y}), \mathcal{M}_2 f_2(\mathbf{y}), \ldots, \mathcal{M}_N f_N(\mathbf{y})]^{\mathrm{T}}$$

Lemma 1. *Consider a 1-D function f such that $\mathcal{E}\{|f(y)|\}$ is finite. If $y = xw$ where w is a multiplicative Gamma distributed random variable, with mean 1 and variance $1/k$, then*

$$\mathcal{E}\{xf(y)\} = \mathcal{E}\{y\mathcal{M}f(y)\} \tag{2}$$

Here, the expectations are taken over the realizations of w.

Proof: Letting $y = wx$, we have

$$\begin{aligned}
\mathcal{E}\{y\mathcal{M}f(y)\} &= \int_0^\infty kwx\,\mathrm{d}w \int_0^1 f(swx)q(w)s^{k-1}\,\mathrm{d}s \\
&= kx\int_0^1 s^{k-1}\,\mathrm{d}s \int_0^\infty wf(swx)q(w)\,\mathrm{d}w \\
&= kx\int_0^1 s^{k-1}\,\mathrm{d}s \int_0^\infty \frac{w}{s^2} f(wx) q\Big(\frac{w}{s}\Big)\,\mathrm{d}w \\
&= kx\int_0^\infty f(wx)\,\mathrm{d}w \int_0^1 \frac{w^k}{s^2}\frac{k^k}{\Gamma(k)}\mathrm{e}^{-kw/s}\,\mathrm{d}s \\
&= kx\int_0^\infty f(wx)\frac{k^k w^k}{\Gamma(k)}\left[\frac{\mathrm{e}^{-kw/s}}{kw}\right]_{0^+}^1\,\mathrm{d}w \\
&= x\int_0^\infty f(wx)\frac{k^k w^{k-1}}{\Gamma(k)}\mathrm{e}^{-kw}\,\mathrm{d}w \\
&= x\int_0^\infty f(wx)q(w)\,\mathrm{d}w = \mathcal{E}\{xf(y)\}
\end{aligned}$$

Here, the hypothesis $\mathcal{E}\{|f(y)|\} < \infty$ was used to validate the integration sign exchanges, and the limit at 0^+. ∎

Using Lemma 1, We derive an expression for the unbiased estimate of the risk as follows:

Theorem 1. *An unbiased estimate (or risk) of the MSE is given by the expression*

$$\mathsf{MURE}(\mathbf{f}) = \frac{1}{N}\left(\frac{k}{k+1}\|\mathbf{y}\|^2 + \|\mathbf{f}(\mathbf{y})\|^2 - 2\mathbf{y}^{\mathrm{T}}\mathcal{M}\mathbf{f}(\mathbf{y})\right). \tag{3}$$

Proof:

Since $\|\mathbf{f}(\mathbf{y}) - \mathbf{x}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{f}(\mathbf{y})\|^2 - 2\mathbf{x}^{\mathrm{T}}\mathbf{f}(\mathbf{y})$ we need to find two functions *of* $\mathbf{y}$ *alone* that are unbiased estimates of $\|\mathbf{x}\|^2$ and $\mathbf{x}^{\mathrm{T}}\mathbf{f}(\mathbf{y})$.

- First, $\mathbf{x}^{\mathrm{T}}\mathbf{f}(\mathbf{y}) = x_1 f_1(\mathbf{y}) + x_2 f_2(\mathbf{y}) + \cdots x_N f_N(\mathbf{y})$: From Lemma 1, we have that

  $$\mathcal{E}\{x_i f_i(\mathbf{y})\} = \mathcal{E}\{y_i \mathcal{M}_i f(\mathbf{y})\}$$

  This shows that

  $$\begin{aligned}
  \mathcal{E}\{\mathbf{x}^{\mathrm{T}}\mathbf{f}(\mathbf{y})\} &= \mathcal{E}\{y_1\mathcal{M}_1 f(\mathbf{y})\} + \mathcal{E}\{y_2\mathcal{M}_2 f(\mathbf{y})\} \\
  &\quad + \cdots \mathcal{E}\{y_N\mathcal{M}_N f(\mathbf{y})\} \\
  &= \mathcal{E}\{\mathbf{y}^{\mathrm{T}}\mathcal{M}\mathbf{f}(\mathbf{y})\}
  \end{aligned}$$

  and so that $\mathbf{y}^{\mathrm{T}}\mathcal{M}\mathbf{f}(\mathbf{y})$ is an unbiased estimate of $\mathbf{x}^{\mathrm{T}}\mathbf{f}(\mathbf{y})$.
- Second, $\|\mathbf{x}\|^2 = x_1^2 + x_2^2 + \cdots x_N^2$: looking at the i^{th} component of $\mathbf{y}$, $y_i = x_i w_i$, we have that $\mathcal{E}\{y_i^2\} = x_i^2\mathcal{E}\{w_i^2\} = x_i^2\big(1 + 1/k\big)$. Hence

  $$x_i^2 = \frac{k}{k+1}\mathcal{E}\{y_i^2\},$$

  and finally $\|\mathbf{x}\|^2 = \dfrac{k}{k+1}\|\mathbf{y}\|^2$

Therefore, the unbiased estimate of MSE is

$$\mathsf{MURE}(\mathbf{f}) = \frac{1}{N}\left(\frac{k}{k+1}\|\mathbf{y}\|^2 + \|\mathbf{f}(\mathbf{y})\|^2 - 2\mathbf{y}^{\mathrm{T}}\mathcal{M}\mathbf{f}(\mathbf{y})\right)$$

∎

IV. Denoising Process

The denoising process that we have employed is as follows: First, we decompose the observed signal into different subbands using undecimated wavelet transform. $\mathbf{D}$ and $\mathbf{R}$ represents a wavelet decmposition and reconstruction matrices satisfying the perfect reconstruction condition $\mathbf{RD} = \mathbf{I}$. Typically $\mathbf{D} = [\mathbf{D_1}^T\mathbf{D_2}^T \ldots \mathbf{D_J}^T]^T$ and $\mathbf{R} = [\mathbf{R_1 R_2} \ldots \mathbf{R_J}]$, $\mathbf{D_j}, \mathbf{R_j} \in \mathcal{R}^{N\times N}$, $j = 1$ to J implement a J-band filterbank of undecimated analysis and synthesis filters.

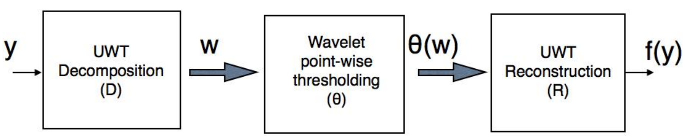


Fig. 1. Denoising process

We suppress noise by proper thresholding of wavelet coefficients in highpass subbands. After thresholding, we apply the corresponding reconstruction filter to revert to the time domain. The key idea of the denoising process is to perform transform-domain denoising by considering the time-domain MURE.

A. Undecimated wavelet decomposition

The wavelet coefficient vector $\mathbf{w}^j = \{w_m^j\}_{m=1}^N$ in the j^{th} subband is given by,

$$\mathbf{w}^j = \mathbf{D_j y},\ j = 1 \text{ to } J,$$

$$\mathbf{D}_j = [d_{m,l}^j]_{1\le m,l\le N},\ \mathbf{R}_j = [r_{m,l}^j]_{1\le m,l\le N}.$$

B. Subband thresholding in the highpass subbands

To suppress the noise, thresholding is performed on the wavelet coefficients using suitable thresholding function in each subband except the lowpass subband.

Since the unbiased estimator given by (3) contains integration term, if we consider soft-thresholding function, the computation of MURE becomes difficult. To reduce computational complexity, we have considered the thresholding function, which reliably approximates the soft-thresholding function. The transfer characteristic of the thresholding function with input w is given by,

$$\theta(w,T) = w\left[1 - \exp\left(-\left(\frac{w}{\gamma}\right)^2\right)\right]$$

where $\gamma = T \times \sqrt{\overline{w}}$ and T, $\overline{w}$ are the thresholding parameter, and subband variance estimator, respectively.

The unbiased estimate of variance of m^{th} wavelet coefficient of j^{th} subband is given by ,

$$\overline{w_m^j} = \frac{1}{k+1}\sum_{l=1}^{N} {d_{m,l}^{j}}^2\, y_l^2.$$

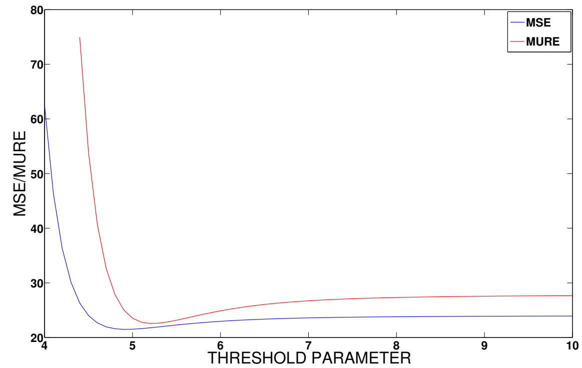


Fig. 2. MSE/MURE (MURE is scaled by $\sim$5) versus thresholding parameter T for *Square* signal with Gamma noise (scale parameter a = 10, shape parameter $\frac{1}{b}$ = 0.1).

The Taylor's series approximation of $\theta(w,T)$ is

$$\theta(w,T) = w\left[\left(\frac{w}{\gamma}\right)^2 - \frac{1}{2!}\left(\frac{w}{\gamma}\right)^4 + \frac{1}{3!}\left(\frac{w}{\gamma}\right)^6 - \cdots\right] \quad (4)$$

The polynomial representation on the right-hand side of (4) makes the computation of the integration term in MURE easier.

C. Wavelet reconstruction

After thresholding, we apply the corresponding undecimated wavelet reconstruction filter, which yields the denoised estimate of original signal x. The denoised estimate $\mathbf{f}(\mathbf{y})$ is,

$$\mathbf{f}(\mathbf{y}) = \sum_{j=1}^{J} \mathbf{R}_j \boldsymbol{\Theta}(\mathbf{w}^j, T),$$

where $\boldsymbol{\Theta}(\mathbf{w}^j, T) = [\theta(w_1^j, T), \theta(w_2^j, T), \cdots, \theta(w_N^j, T)]^T$. The parameter of the thresholding function is T. We choose the optimal T such that MURE is minimized.

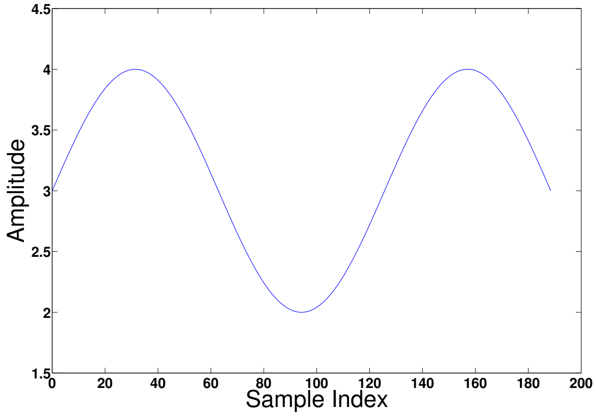
V. Results and Performance Comparison

In simulations, we used Haar wavelet basis and four-level wavelet decomposition. We find the optimal thresholding parameter T that minimizes the unbiased risk estimator (MURE) by exhaustive search method.

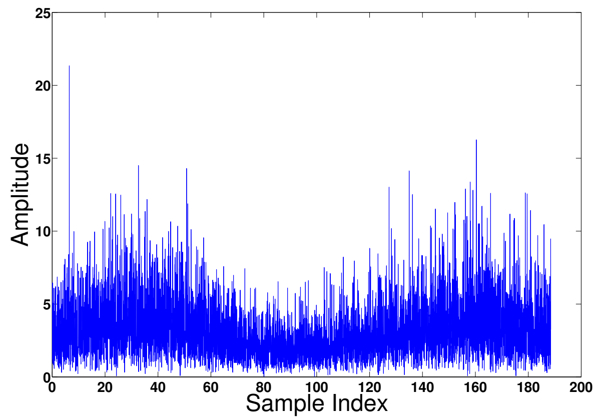
Simulations are performed on different standard signals with Gamma distributed multiplicative noise with varying scale parameter. Our experiments showed that the optimization of MURE gives minima close to those obtained by optimization of MSE. From Figure 2, we observe that although MURE is not so close to the oracle MSE for some values of the threshold parameter T, since MURE minima is so close to oracle MSE minima with respect to threshold parameter, minimization of MURE is nearly equivalent to the minimization of oracle MSE.

Figures 3, 4, and 5 show the noisy signals and corresponding restored signals using MURE-denoising method.

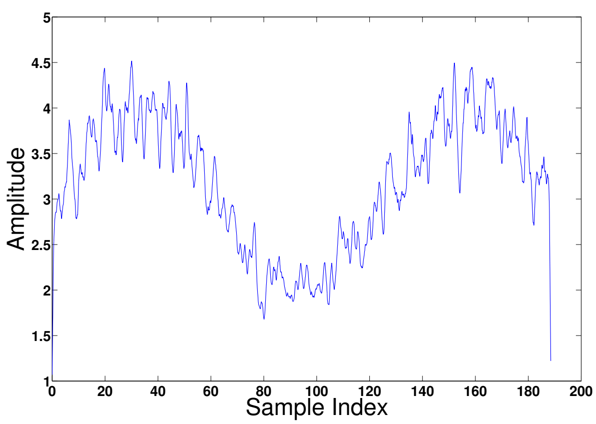
We compare the results obtained by the proposed method with a state-of-the-art method. We compare the performance



(a) Original signal

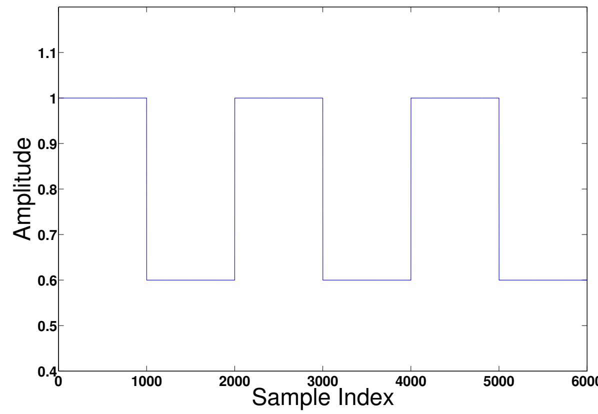


(b) Noisy signal (SNR = 4.84 dB)

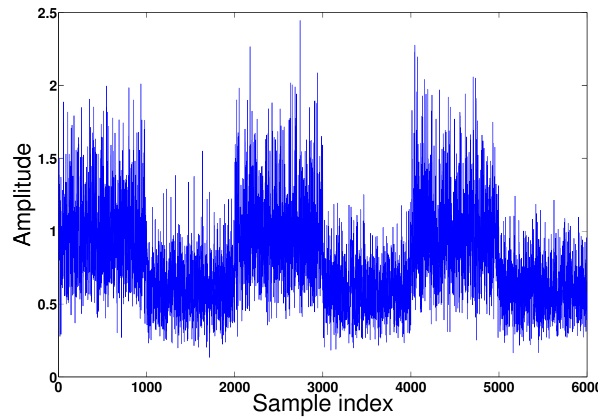


(c) Restored signal (SNR = 20.86 dB)

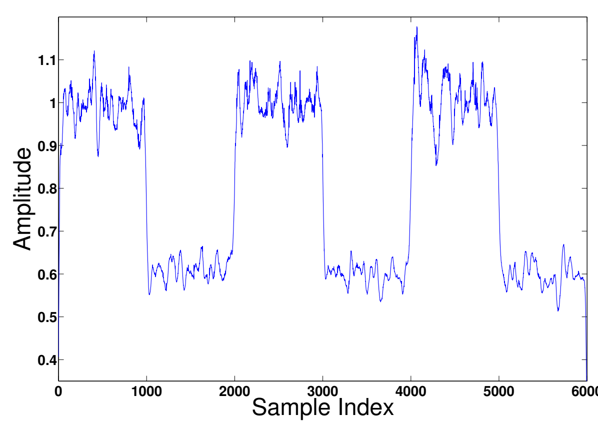
Fig. 3. Denoising performance on a *sinusoidal* signal



(a) Original signal



(b) Noisy signal (SNR = 10.07 dB)



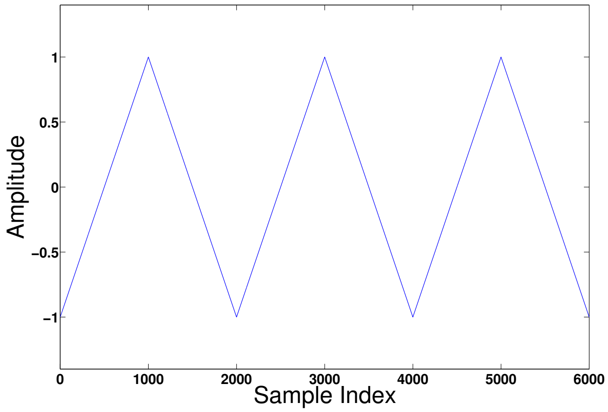
(c) Restored signal (SNR = 24.17 dB)

Fig. 4. Denoising performance on a *square* wave

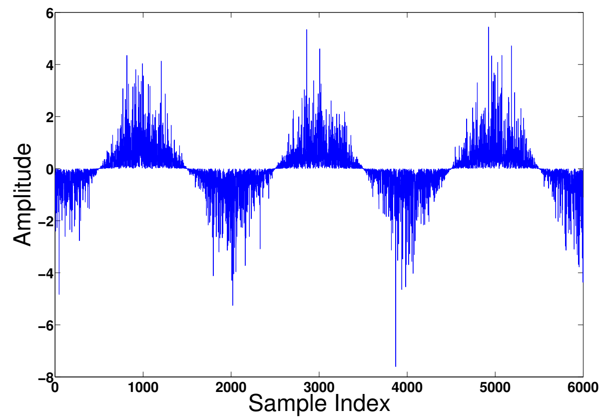
of the proposed denoising method on noisy sinusoidal signal, square signal and triangular signal with the variational approach [8]. Table I shows a comparison of results obtained by the proposed method with the variational approach. The SNR improvement for the proposed method is competitive with the state-of-the-art method.

VI. Conclusions

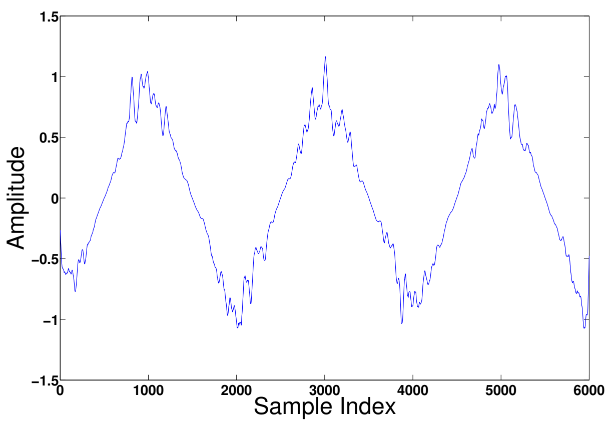
We developed a new risk estimation framework for suppressing multiplicative noise in one-dimensional signals. Starting with the mean-square error as the risk function, we have developed an unbiased estimator that approximates the risk and enables efficient optimization of the denoising function. The



(a) Original signal



(b) Noisy signal (SNR = 0.07 dB)



(c) Restored signal (SNR = 15.88 dB)

Fig. 5. Denoising performance on a *triangular* wave

result has been derived specifically for the case of Gamma distributed noise. However, suitable risk estimators may be derived for other distributions as well. We have considered undecimated wavelet decomposition based denoising, where the denoising function parameters on every subband are optimized for using the risk estimator. Performance analysis on noisy 1-D signals showed that the proposed method is capable of enhancing the signal-to-noise ratio significantly. Comparisons with a state-of-the-art technique showed that the proposed technique has about 0.5 to 1.5 dB higher output SNR. A thorough comparison with other techniques, and performance assessment on real signals is currently being carried out.

Signal	SNR_{in}(dB)	SNR_{out}(dB) [8]	SNR_{out}(dB) MURE
sinusoidal	4.69	19.76	21.03
sinusoidal	7.20	21.70	23.14
sinusoidal	9.98	24.13	25.24
sinusoidal	14.75	27.41	28.15
square	4.90	19.56	21.06
square	6.85	20.42	22.39
square	9.90	23.36	24.52
square	14.63	27.82	27.46
triangular	4.76	18.25	20.43
triangular	7.0	20.61	22.16
triangular	9.90	23.03	23.24
triangular	12.89	25.44	24.73

TABLE I
COMPARISON OF THE PROPOSED DENOISING METHOD WITH THE VARIATIONAL APPROACH [8].

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