Fresnelets—a new wavelet basis for digital holography

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ABSTRACT

We present a new class of wavelet bases—Fresnelets—which is obtained by applying the Fresnel transform operator to a wavelet basis of L_2. The thus constructed wavelet family exhibits properties that are particularly useful for analyzing and processing optically generated holograms recorded on CCD-arrays.

We first investigate the multiresolution properties (translation, dilation) of the Fresnel transform that are needed to construct our new wavelet. We derive a Heisenberg-like uncertainty relation that links the localization of the Fresnelets with that of the original wavelet basis. We give the explicit expression of orthogonal and semi-orthogonal Fresnelet bases corresponding to polynomial spline wavelets. We conclude that the Fresnel B-splines are particularly well suited for processing holograms because they tend to be well localized in both domains.

Keywords: Fresnelets, Digital holography, Fresnel transform, B-splines, wavelets.

1. INTRODUCTION

Wavelets are a powerful tool for signal processing and analysis. They are especially useful for data compression and for detecting and characterizing signal singularities. Here, we present a new family of wavelet bases that is tailored to the specificities of digital holography.

Digital holography[1] is an imaging method in which a hologram[2] is recorded with a CCD-camera and reconstructed numerically. The hologram results from the interference between the wave reflected or transmitted by the object to be imaged and a reference wave. One arrangement that is often used is to record the distribution of intensity in the hologram plane at the output of a Michelson interferometer. The digital reconstruction of the complex wave (amplitude and phase) near the object is based on the Fresnel transform, an approximation of the diffraction integral.[3]

Since it is in essence a lensless process, digital holography tends to spread out sharp details like object edges over the entire image plane. Therefore, standard wavelets, which are typically designed to process piecewise smooth signals, will give poor results when applied directly to the hologram.

While analytical solutions to the diffraction problem can be given in terms of Gauss-Hermite functions, those do not satisfy the completeness requirements[4] of wavelet theory and are therefore of limited use for digital processing. Instead, we chose to work with B-spline associated bases which have many advantages.[5] Also, B-splines asymptotically tend to gaussians as their degree increases, while fulfilling the partition of unity relation which is essential for the closure of L_2.[4]

2. UNITARY FRESNEL TRANSFORM

2.1. Definitions

We define the unitary Fresnel transform with parameter τ of a function $f \in L_2(\mathbb{R}, \mathbb{C}, \mathrm{d}x)$ as the convolution integral:

$$\widetilde{f}_\tau(x) = (f * k_\tau)(x) \text{ with } k_\tau(x) = \frac{1}{\tau}\,\mathrm{e}^{i\pi(\frac{x}{\tau})^2} \tag{1}$$

which is well defined in the L_2 sense.
The frequency response of this operator is:

$$\widehat{k}_\tau(\nu) \quad = \quad \mathrm{e}^{i\frac{\pi}{4}}\,\mathrm{e}^{-i\pi(\tau\nu)^2}, \tag{2}$$

with the property that $\left|\widehat{k}_\tau(\nu)\right| = 1$, $\forall \nu \in \mathbb{R}$. As the transform is unitary, we get a Parseval equality:

$$\forall f, g \in L_2 \qquad < f, g > = < \widetilde{f}_\tau, \widetilde{g}_\tau > . \tag{3}$$

The inverse transform in the space domain is given by:

$$f(x) = (\widetilde{f}_\tau * k_\tau^{-1})(x) \text{ with } k_\tau^{-1}(x) = k_\tau^*(x) = \frac{1}{\tau} \,\mathrm{e}^{-i\pi(\frac{x}{\tau})^2}. \tag{4}$$

It is simply derived by inverting the operator in the Fourier domain:

$$\widehat{k}_\tau^{-1}(\nu) = \mathrm{e}^{-i\frac{\pi}{4}} \,\mathrm{e}^{i\pi(\tau\nu)^2} = \widehat{k}_\tau^*(\nu). \tag{5}$$

2.2. Properties

For building our new wavelet family, it is essential to understand how the Fresnel transform behaves with respect to key operations in multiresolution wavelet theory such as dilation and translation. These fundamental properties are summarized in the following table.

Property	**Function**	**Transformed Function**
	$f(x)$	$\widetilde{f}_\tau(x)$
Duality	$(\widetilde{f}_\tau)^*(x)$	$f^*(x)$
Translation	$f(x - x_0)$	$\widetilde{f}_\tau(x - x_0)$
Dilation	$f(\frac{x}{s})$	$\widetilde{f}_{\frac{\tau}{s}}(\frac{x}{s})$

2.3. Two-dimensional Fresnel Transform

We define the unitary two-dimensional Fresnel transform with parameter τ of a function $f \in L_2(\mathbb{R}^2, \mathbb{C}^2, \mathrm{d}x^2)$ as the 2D convolution integral:

$$\widetilde{f}_\tau(\vec{x}) = \widetilde{f}_\tau(x, y) = (f * K_\tau)(\vec{x}) \tag{6}$$

where the separable kernel is :

$$K_\tau(\vec{x}) = \frac{1}{\tau^2} \,\mathrm{e}^{i\pi(\frac{\|\vec{x}\|}{\tau})^2} = k_\tau(x) \cdot k_\tau(y) \tag{7}$$

Thus, we will be able to perform most of our mathematical analysis in one dimension and simply extend the results to two dimensions by using separable basis functions.

The two-dimensional unitary Fresnel transform is linked to the diffraction problem in the following manner. Consider a complex wave travelling in the z-direction. We denote by $\psi(x, y)$ the complex amplitude of the wave at distance $z = 0$ and by $\Psi(x, y)$ the diffracted wave at a distance $z = d$. If the requirements for the Fresnel approximation are fulfilled, we have that:

$$\begin{aligned} \Psi(x, y) &= \frac{\mathrm{e}^{ikd}}{i\lambda d} \iint \psi(\xi, \eta) \,\mathrm{e}^{\frac{i\pi}{\lambda d}\left((\xi - x)^2 + (\eta - y)^2\right)} \mathrm{d}\xi \mathrm{d}\eta \\ &= -i \,\mathrm{e}^{ikd} \,\widetilde{\psi}_{\sqrt{\lambda d}}(x, y) \end{aligned}$$

where λ is the wavelength of the light and $k = 2\pi/\lambda$ its wavenumber.

3. THE FRESNEL TRANSFORM AND LOCALIZATION

Perhaps one of the least intuitive aspects of holography is that the diffraction process tends to spread out features that are initially well localized in the object domain. Getting a better understanding of the notion of resolution in holography is what we are after in this section.

3.1. Link with the Fourier Transform

So far, we have considered the Fresnel transform as a convolution operator. In order to derive a Heisenberg-like uncertainty relation for this transform, it is useful to connect it to the Fourier transform. The Fresnel transform $\widetilde{g}_\tau$ of a function $g \in L_2(\mathbb{R}, \mathbb{C}, \mathrm{d}x)$ can be expressed as[3]:

$$\widetilde{g}_\tau(x) = k_\tau(x) \cdot \widehat{f}(\frac{x}{\tau^2}), \tag{8}$$

where

$$f(x) = \tau k_\tau(x) g(x). \tag{9}$$

3.2. Uncertainty relation

We denote the mean μ_f of a function f by:

$$\mu_f = \frac{1}{\|f\|^2} \int_{-\infty}^{\infty} t|f(t)|^2 \,\mathrm{d}t \tag{10}$$

and its variance σ_f^2 by:

$$\sigma_f^2 = \frac{1}{\|f\|^2} \int_{-\infty}^{\infty} (t - \mu_f)^2 |f(t)|^2 \,\mathrm{d}t. \tag{11}$$

We recall the Heisenberg uncertainty relation for the Fourier transform.

THEOREM 3.1 (HEISENBERG UNCERTAINTY RELATION – FOURIER TRANSFORM). *Let $f \in L_2(\mathbb{R}, \mathbb{C}, dx)$. We have following inequality:*

$$\sigma_f^2 \, \sigma_{\widehat{f}}^2 \geq \frac{1}{16\pi^2}. \tag{12}$$

This inequality is an equality if and only if there exist $(u, \xi, a, b) \in \mathbb{R}^2 \times \mathbb{C} \times \mathbb{R}$ such that:

$$f(t) = a e^{i\xi t} e^{-b(t-u)^2}. \tag{13}$$

We can prove a similar expression for the unitary Fresnel Transform:

LEMMA 3.2 (UNCERTAINTY RELATION FOR THE FRESNEL TRANSFORM). *Let $g \in L_2(\mathbb{R}, \mathbb{C}, dx)$. We have following inequality:*

$$\sigma_g^2 \sigma_{\widetilde{g}_\tau}^2 \geq \frac{\tau^4}{16\pi^2}. \tag{14}$$

This inequality is an equality if and only if there exist $(u, \xi, a, b) \in \mathbb{R}^2 \times \mathbb{C} \times \mathbb{R}$ such that:

$$g(x) = a e^{i\xi x} e^{-b(x-u)^2} e^{-i\pi(\frac{x}{\tau})^2}. \tag{15}$$

If $\widehat{g} \in L_2(\mathbb{R}, \mathbb{R}, dx)$, the following relation holds:

$$\sigma_{\widetilde{g}_\tau}^2 \sigma_g^2 = \sigma_g^4 + \tau^4 \sigma_{\widehat{g}}^2 \sigma_g^2 \geq \sigma_g^4 + \frac{\tau^4}{16\pi^2}. \tag{16}$$

The previous results suggests that Gaussians and Gabor-like functions, modulated with the kernel function as in (15) should be well suited for processing and reconstructing holograms.

4. FRESNELET BASES

Instead of using Gabor-like functions which cannot yield a stable basis of L_2,[6] we will base our construction on B-splines. These are Gaussian-shaped functions that can easily yield wavelets which are well localized in the sense of the uncertainty principle of equation (16).[7]

4.1. B-splines

The central B-spline of degree n is constructed from the $(n+1)$-fold convolution of a rectangular pulse:

$$\beta^n(x) = \underbrace{\beta^0 * \cdots * \beta^0}_{n+1 \text{ times}}(x) \tag{17}$$

$$\beta^0(x) = \begin{cases} 1, & |x| < \frac{1}{2} \\ \frac{1}{2}, & |x| = \frac{1}{2} \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

Another equivalent definition is:

$$\beta^n(x) = \Delta^{n+1} * \frac{(x)_+^n}{n!} \tag{19}$$

where Δ^{n+1} is the $(n+1)$th centered finite-difference operator:

$$\Delta^{n+1} = \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \delta(x + \frac{n+1}{2} - k) \tag{20}$$

and where $(x)_+^n = \max(0, x)^n$ is the one-sided power function. This gives explicitly:

$$\beta^n(x) = \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \frac{(x + \frac{n+1}{2} - k)_+^n}{n!}. \tag{21}$$

4.2. Polynomial spline wavelets

The B-splines satisfy all the requirements of a valid scaling function of L_2: (a) they form a Riesz basis, (b) they satisfy a two scale relation and (c) they fulfill the partition of unity condition. Thus, they can be used to generate a multiresolution analysis of L_2. Unser *et al.*[8] have shown that one can construct a general family of semi-orthogonal spline wavelets of the form:

$$\psi^n({x}/{2}) = \sum_k g(k) \beta^n(x - k) \tag{22}$$

such that the functions

$$\left\{ \psi_{i,k}^n = 2^{-{i}/{2}} \psi(2^{-i}x - k) \right\}_{i \in \mathbb{Z}, k \in \mathbb{Z}} \tag{23}$$

form a Riesz basis of L_2. These wavelets come in different brands: orthogonal, B-spline (of compact support), interpolating, etc ... They are all linear combinations of B-splines and are thus entirely specified from the sequence $g(k)$ in equation (22).

The main point here is that by using the properties of the Fresnel transform (linearity, shift-invariance and scaling), we can easily derive the family of functions $\{\widetilde{\psi}_{i,k,\tau}^n\}$, provided that we know the Fresnel transform of their main constituent, the B-spline.

4.3. Fresnelets

Since the Fresnel transform is an unitary operator from L_2 into L_2, it maps Riesz bases into other Riesz bases, with the same Riesz bounds.

Let $\{\psi^k\}_{k \in \mathbb{Z}}$ be a Riesz basis of L_2 and $\{\mathring{\psi}^k\}_{k \in \mathbb{Z}}$ its dual. By applying the Fresnel transform to each basis function, we construct the Fresnelet basis $\{\widetilde{\psi}_\tau^k\}_{k \in \mathbb{Z}}$ and its dual $\{\mathring{\widetilde{\psi}}_\tau^k\}_{k \in \mathbb{Z}}$. The Parseval relation (3) implies $\forall f \in L_2$:

$$f = \sum_k < f, \mathring{\psi}^k > \psi^k = \sum_k < \widetilde{f}_\tau, \mathring{\widetilde{\psi}}_\tau^k > \psi^k. \tag{24}$$

This relation is most relevant for the reconstruction of an image f given its transform $\widetilde{f}$. It suffices to compute the series of inner products $< \widetilde{f}_\tau, \overset{\circ}{\widetilde{\psi}}{}^k_\tau >$; these can then be used directly as expansion coefficients in the image domain. A potential advantage of such a hologram domain approach is that it may be less sensitive to boundary effects than the traditional DFT-based technique.

4.4. Fresnel transform of a B-spline

We have seen that we can construct orthogonal and semi-orthogonal bases from linear combination of B-splines. Furthermore, we have just pointed out that these bases map into new bases via the unitary Fresnel transform. Therefore, to complete the characterization of our new wavelets, the Fresnelet basis, we need to calculate the Fresnel transform of a B-spline.

Instead of the one-sided power function used in (21) we will work with the function:

$$u_{n,\tau}(x) = \int_0^x \frac{(x-\xi)^n}{n!}\, k_\tau(\xi)\, \mathrm{d}\xi \tag{25}$$

which satisfies:

$$\frac{\partial}{\partial x} u_{n,\tau}(x) = \begin{cases} u_{n-1,\tau}(x), & \text{for } n > 0 \\ k_\tau(x), & \text{for } n = 0. \end{cases} \tag{26}$$

We can now calculate the Fresnel transform of a B-spline of degree n: $\beta^n(x)$, which we call a Fresnel B-spline of degree n and denote by $\widetilde{\beta}^n_\tau(x)$:

$$\begin{aligned} \widetilde{\beta}^n_\tau(x) &= (\beta^n * k_\tau)(x) \\ &= (\beta^n * u^{(n+1)}_{n,\tau})(x) \\ &= \left(\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\beta^n\right) * u_{n,\tau}(x) \\ &= \left(\sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \delta(x-k)\right) * u_{n,\tau}(x) \\ &= \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} u_{n,\tau}(x-k). \end{aligned} \tag{27}$$

The $u_{n,\tau}$ can be calculated recursively as:

$$u_{n,\tau}(x) = \frac{\tau}{2i\pi n!} x^{n-1} - \frac{\tau^2}{2i\pi n} u_{n-2,\tau}(x) + \frac{x}{n} u_{n-1,\tau}(x). \tag{28}$$

For $n = 0$ we have:

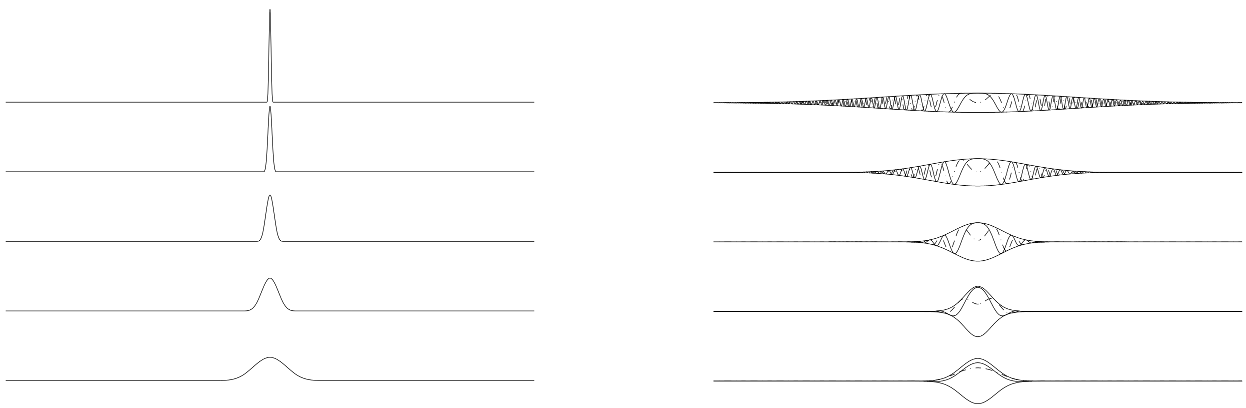
$$u_{0,\tau}(x) = \frac{1}{\sqrt{2}} \left(C(\frac{\sqrt{2}}{\tau}x) + i\, S(\frac{\sqrt{2}}{\tau}x) \right) \tag{29}$$

where $C(x)$ and $S(x)$ are the *Fresnel integrals*:

$$C(x) = \int_0^x \cos(\frac{\pi}{2}t^2)\, \mathrm{d}t, \quad S(x) = \int_0^x \sin(\frac{\pi}{2}t^2)\, \mathrm{d}t. \tag{30}$$

For $n = 1$ we have:

$$u_{1,\tau}(x) = x\, u_{0,\tau}(x) - \frac{\tau^2}{2i\pi}(k_\tau(x) - \frac{1}{\tau}). \tag{31}$$



(a) (b)

Figure 1. B-Spline multiresolution and its Fresnel counterpart. (a) B-splines: $2^{-k/2}\beta^3(2^{-k}x)$, $k = 0, 1, 2, 3, 4$. (b) Corresponding Fresnel B-splines: $2^{-k/2}\widetilde{\beta}^3_{\tau 2^{-k}}(2^{-k}x)$. In this experiment, $\tau = 0.9$.

5. EXAMPLE

In Figure 1 we show a sequence of dyadic scaled B-splines of degree $n = 3$ and its counterpart in the Fresnelet domain. The effect of the spreading is clearly visible. This has following consequence: if one wants to reconstruct a hologram at a fine scale, that is, express it as a sum of narrow B-splines, the equivalent basis functions on the hologram get larger. Our special choice of Fresnelet bases limits this phenomenon as much as possible; it is nearly optimal in the sense of our uncertainty relation for functions with real fourier transforms.

6. CONCLUSION

We have investigated the properties of the Fresnel transform, in particular those linked to wavelets (translation and dilation). We derived a Heisenberg-like uncertainty relation that gives a bound on the spatial spreading in the Fresnel domain (hologram). This guided us in the choice and definition of a new wavelet—Fresnelet—that should exhibit similar spatial spreading as Gaussians and Gabor-like atoms while exhibiting the fundamental multiresolution and orthogonality requirements of wavelet theory. The thus obtained class of wavelet bases were obtained by applying the Fresnel transform operator to polynomial spline wavelets. We believe that understanding the multiresolution structure and the form of these new wavelet-like bases should be useful for deriving new numerical schemes for holographic reconstruction.

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