# OPTIMAL INTERPOLATION OF FRACTIONAL BROWNIAN MOTION GIVEN ITS NOISY SAMPLES

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### ABSTRACT

We consider the problem of estimating a fractional Brownian motion known only from its noisy samples at the integers. We show that the optimal estimator can be expressed using a digital Wiener-like filter followed by a simple time-variant correction accounting for nonstationarity.

Moreover, we prove that this estimate lives in a symmetric fractional spline space and give a practical implementation for optimal upsampling of noisy fBm samples by integer factors.

## 1. INTRODUCTION

Natural signals are usually not stationary, but often present some degree of statistical scale invariance, at least over a significant scale range. An interesting class of stochastic processes that exhibit such fractal behavior is fractional Brownian motion (fBm) [1, 2]. The defining properties of fBm are: 1) full statistical scale invariance; 2) stationarity of the increments; and 3) global Gaussian statistics.

When a signal is Gaussian and stationary, the optimal method for estimating it from its (Gaussian stationary) noisy uniform samples, is to filter these samples (digitally) and interpolate them in the function space generated by the integer shifts of the autocorrelation of the underlying process [3].

Unfortunately, in the case of non-stationary signals such as fBm, this approach is not valid. The goal of this paper is to derive the optimal estimator for such processes and to investigate the extent to which the classical signal processing techniques are still applicable.

We will see that the full denoising process involves a stationary part—digital filtering and interpolation in a fractional spline space—and a non-stationary part which, in particular, ensures that the estimate vanishes at $t = 0$, just like an fBm [4]. Interestingly, we will establish that the best estimation of an fBm given its noisy samples is a fractional spline of degree $2\gamma$ where $\gamma$ is the Hurst exponent of the fBm. In particular, we will recover a classical result by Paul Lévy that states that the best interpolation (noiseless case) of the usual Brownian motion ($\gamma = 1/2$) is piecewise-linear [5].

## 2. CHARACTERIZATION OF FRACTIONAL BROWNIAN MOTIONS

An fBm $B_\gamma(t)$ with Hurst exponent $0 < \gamma < 1$ is a zero-mean Gaussian process whose correlation is given by:

$$\mathscr{E}\{B_\gamma(t)B_\gamma(t')\} = \frac{C_\gamma}{2}\left(|t|^{2\gamma} + |t'|^{2\gamma} - |t-t'|^{2\gamma}\right) \quad (1)$$

where $\mathscr{E}\{\cdot\}$ is the expectation operator and $C_\gamma$ a positive constant [1]. It is the unique scale invariant[1] Gaussian process whose variogram $\mathscr{E}\{|B_\gamma(t)-B_\gamma(t')|^2\}$ is a function of $(t-t')$ alone [6]. A direct consequence of (1) is $\mathscr{E}\{B_\gamma(0)^2\} = 0$, which implies that $B_\gamma(0) = 0$ almost surely—i.e., with probability one.

An fBm can also be expressed as a stochastic integral (that can be understood either in the Itô formulation [7], or in the sense of generalized stochastic processes of Gel'fand and Vilenkin [8]) through the use of a normalized[2] Gaussian white noise process $W(\omega)$ [9]:

$$B_\gamma(t) = \frac{\varepsilon_\gamma^2}{\sqrt{2\pi}}\int \frac{e^{j\omega t}-1}{|\omega|^{\gamma+1/2}} W(\omega)\,\mathrm{d}\omega,$$

where $\varepsilon_\gamma^2 = \Gamma(2\gamma+1)\sin(\pi\gamma)\,C_\gamma$.

This expression indicates that an fBm essentially behaves like a stationary process whose PSD would be $\varepsilon_\gamma^2/|\omega|^{2\gamma+1}$. More specifically, we can prove that the integer samples of an fBm are whitened by the digital filter whose discrete-time Fourier transform is

$$\frac{\left|2\sin\frac{\omega}{2}\right|^{\gamma+1/2}}{\varepsilon_\gamma\sqrt{A^{\gamma-1/2}(e^{j\omega})}},$$

where $A^{\gamma-1/2}(e^{j\omega})$ is the autocorrelation filter of a fractional spline of degree $\gamma - 1/2$, the expression of which is

[1] The rescaled stochastic process $B_\gamma(at)$ follows the same probability law as the process $a^\gamma B_\gamma(t)$ for any positive real $a$.

[2] i.e., $\mathscr{E}\{W(t)W(t')\} = \delta(t-t')$.

given in the next section—see (5). Consequently, we can generate a good approximation of the samples of an fBm by filtering a discrete white noise with the inverse of the above filter.

## 3. SYMMETRIC FRACTIONAL B-SPLINES

Given uniform samples $f(n)$ of a continuous-time function $f(t)$, it is customary to built an interpolated version of these samples that would be close to $f(t)$ using a Shannon-like expression

$$f_{\mathrm{est}}(t) = \sum_{k\in\mathbb{Z}} c_n \varphi(t-k) \tag{2}$$

where $\varphi(t)$ is some basis function (such as the sinc-function[3]) and the coefficients $c_n$ are chosen so as to fit the samples (interpolation condition): i.e., $f_{\mathrm{est}}(k) = f(k)$ [10]. When the samples are noisy, it is better to relax the interpolation condition and use a quasi-interpolation instead. The main point however is that (2) remains valid.

B-splines are an attractive choice for $\varphi(t)$ because of their excellent approximation and multiresolution properties [10]. A superset of the natural B-splines has been defined in [11]. Here, we will be concerned only with symmetric fractional B-splines which can be defined as:

$$\beta_*^\alpha(t) = \int_{-\infty}^{\infty} |\operatorname{sinc} F|^{\alpha+1} e^{j2\pi Ft}\,\mathrm{d}F. \tag{3}$$

Indeed, we will see in the next section that, when $f(t)$ is an fBm with Hurst exponent $\gamma$, then the optimal quasi-interpolation of $f(t)$ given its noisy samples can be expressed as (2) with $\varphi(t) = \beta_*^{2\gamma}(t)$.

Symmetric fractional B-splines satisfy an $M$-scale difference equation (for any integer $M \geq 2$) which is easily expressed in Fourier variables:

$$\hat\beta_*^\alpha(\omega) = \left|\frac{\sin\frac{\omega}{2}}{M\sin\frac{\omega}{2M}}\right|^{\alpha+1} \hat\beta_*^\alpha\Big(\frac{\omega}{M}\Big). \tag{4}$$

This property is especially useful for defining fast multiscale algorithms (wavelet transforms, scale changes) as we will see in the sequel.

Working with B-splines involves manipulating spline autocorrelation sequences defined by their discrete-time Fourier transform

$$A^\alpha(e^{j\omega}) = \sum_{k\in\mathbb{Z}} \left|\hat\beta_*^\alpha(\omega+2k\pi)\right|^2 \tag{5}$$

for which a fast computation is available [12].

[3] By definition here, $\operatorname{sinc} x = \dfrac{\sin \pi x}{\pi x}$.

## 4. OPTIMAL ESTIMATION OF AN FBM

Our problem of interest is the estimation of an fBm $B_\gamma(t)$ from a series of noisy samples $y_k = B_\gamma(k) + N(k)$ for all $k \in \mathbb{Z}$ where $\{N(k)\}_{k\in\mathbb{Z}}$ is a Gaussian stationary discrete noise, independent from $B_\gamma(t)$. The best estimate can be expressed formally in a Bayesian setting as

$$B_{\gamma,\mathrm{est}}(t) = \mathscr{E}\big\{B_\gamma(t)\big|\{y_k\}_{k\in\mathbb{Z}}\big\}.$$

It is the one that minimizes the expected square error $\mathscr{E}\{|B_\gamma(t) - B_{\gamma,\mathrm{est}}(t)|^2\}$.

We denote by $r_k$ the autocorrelation sequence of the noise; i.e., $r_k = \mathscr{E}\{N(k)N(0)\}$ and by $R(e^{j\omega}) = \sum_k r_k e^{-jk\omega}$ its discrete-time Fourier transform.

**Theorem 1** *The optimal estimate of $B_\gamma(t)$, given $y_k = B_\gamma(k) + N(k)$, is a fractional spline of degree $2\gamma$:*

$$B_{\gamma,\mathrm{est}}(t) = \sum_{k\in\mathbb{Z}} c_k \beta_*^{2\gamma}(t-k) \tag{6}$$

*where the coefficients $c_k$ are given by:*

$$c_k = h_k * y_k - \lambda\, h_k * r_k. \tag{7}$$

*The scalar quantity $\lambda$ and the filter $H(z) = \sum_{k\in\mathbb{Z}} h_k z^{-k}$ are specified by*

$$H(e^{j\omega}) = \frac{1}{\left|2\sin\frac{\omega}{2}\right|^{2\gamma+1}\dfrac{R(e^{j\omega})}{\varepsilon_\gamma^2} + A^{\gamma-1/2}(e^{j\omega})}$$

$$\lambda = \frac{\sum_k h_k * y_k\, \beta_*^{2\gamma}(k)}{\sum_k h_k * r_k\, \beta_*^{2\gamma}(k)}$$

The proof is somewhat technical and will be given elsewhere [4]. This result shows that the optimal estimation of the non-stationary signal $B_\gamma(t)$ is not a mere filtered version of the $y_k$'s, as it would be the case in the stationary case. However, the difference is not so large because, as is apparent from (7), the first term expressing $c_k$ is indeed a filtered version of the measurements $y_k$. The presence of the second term, on the other hand, is a consequence of non-stationarity, and, in particular, of the fact that $B_\gamma(t)$ is known to vanish at $t = 0$.

Note however, that if we, quite reasonably, assume that the autocorrelation of the noise is localized around 0 (in the limit white noise case, we would even have $r_k = 0$ for all $k \neq 0$) then this second term is guaranteed to vanish when $k \to \infty$ because $h_k$ is itself a localized filter [4]. Thus, for large values of $t$, the best estimate of an fBm is simply a filtered version of its noisy samples, as in the stationary case.

A side result of this theorem is the quality of the estimation, which can be expressed as the expectation of the square error between the fBm and its estimate.

**Theorem 2** *The minimal expected square error between $B_\gamma(t)$ and its estimation based on the noisy samples $y_k = B_\gamma(k) + N(k)$ is given by*

$$\mathscr{E}\{|B_\gamma(t) - B_{\gamma,\mathrm{est}}(t)|^2\} = \sigma_0^2(t) - \rho^2(t)\sigma_0^2(0) \qquad (8)$$

*where*[4]

$$\sigma_0^2(t) = \frac{C_\gamma}{2} \sum_{k\in\mathbb{Z}} |t-k|^{2\gamma} h_k * \beta_*^{2\gamma}(t-k)$$

$$\rho(t) = \frac{\sum_k h_k * r_k\, \beta_*^{2\gamma}(t-k)}{\sum_k h_k * r_k\, \beta_*^{2\gamma}(k)}$$

## 5. IMPLEMENTATION—EXAMPLE

Let $M$ be some integer greater than one. We present here an algorithm for estimating $B_\gamma(t)$ at the sampling times $t = n/M$ given its noisy samples $B_\gamma(k) + N(k)$ with the same hypotheses as in Section 4. From Theorem 1 we know that the optimal estimate of $B_\gamma(nM)$ are $B_{\gamma,\mathrm{est}}(n/M)$. Based on (6), the system that transforms $c_k$ into $B_{\gamma,\mathrm{est}}(n/M)$ is thus an upsampler by $M$ followed by filtering by $G_M(z) = \sum_k \beta_*^{2\gamma}(k/M) z^{-k}$.

By using the $M$-scale relation (4), it is a simple matter to show that

$$G_M(e^{j\omega}) = M \left| \frac{\sin \frac{M\omega}{2}}{M \sin \frac{\omega}{2}} \right|^{2\gamma+1} A^{\gamma-1/2}(e^{j\omega}).$$

The full estimation process indicating how to compute $B_{\gamma,\mathrm{est}}(k)$ from $y_k = B_\gamma(k) + N(k)$ is shown in Fig. 1.

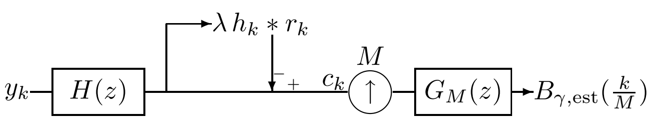


**Fig. 1**. $M$-upsampling algorithm of a noisy fBm of Hurst exponent $\gamma$. See text for the definition of the expressions defining $H(z)$, $\lambda$, $r_k$ and $G_M(z)$.

Since it is inpractical to assume an infinite nontrivial sequence of samples, the implementation of this scheme has to make an approximation, namely that the missing samples can be obtained by periodic repetition of the known samples (periodic boundary conditions). Then, the actual implementation of all the filtering operations can be done using a discrete Fourier transform.

An example of processing applied to a simulated fBm, generated as specified in Section 2, is given in Fig. 2.

[4]For the sake of simplicity, we have written $h_k * \beta_*^{2\gamma}(t-k)$ to mean $\sum_l h_l \beta_*^{2\gamma}(t-(k-l))$

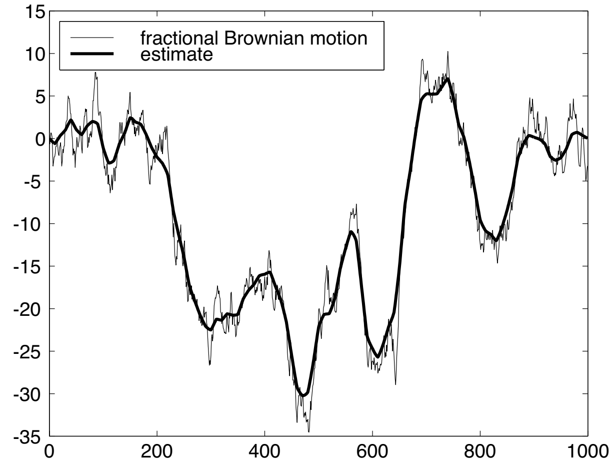


**Fig. 2**. Optimal upsampling by $M = 10$ of an fBm with Hurst exponent $\gamma = 0.6$ ($\varepsilon_\gamma = 1$) corrupted by discrete white noise of unit energy.

## 6. CONCLUSION

We have given a theoretical result establishing that a fractional spline of degree $2\gamma$ is the best estimator for a fractional Brownian motion of Hurst exponent $\gamma$, given its noisy samples. What is especially interesting here is that, although the expression of the coefficients of the spline expansion is time-variant due to nonstationarity of the process, these can still be computed efficiently by filtering. The detailed expression of the estimate indicates that, away from $t = 0$, an fBm actually behaves like a stationary process.

## 7. REFERENCES

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