LOCALISING DIFFUSION SOURCES FROM SAMPLES TAKEN ALONG UNKNOWN PARAMETRIC TRAJECTORIES

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ABSTRACT

In a recent paper we showed that it is possible to localise diffusion sources observed with a mobile sensor whilst simultaneously estimating the piecewise linear trajectory of the sensor. Here we address the case in which the sensor moves along an arbitrary unknown parametric trajectory and we show that by solving a linear system of equations, we can retrieve the inner products between the parameters of the trajectory. From these inner products we then retrieve the curve parameters up to an orthogonal transformation, which allows us to also perfectly estimate the amplitudes of the sources and find their locations up to an orthogonal transformation.

Index Terms— Diffusion field, sampling, localisation, trajectory estimation, diffusion SLAM.

1. INTRODUCTION

Sampling and reconstruction of physical fields has a wide range of real-life applications, such as finding nuclear and chemical leakages [1], detection of wild fires or pollution sources [2] or localising neuronal source activities from electroencephalographic (EEG) signals [3]. This problem has attracted significant interest in the signal processing community [4–9]. With the exception of [9], most of the state-of-the-art methods assume that the diffusion field is monitored using sensors at fixed locations, which are known. Sampling physical fields along trajectories using mobile sensors was considered in [10, 11], but assuming the trajectories are known.

The problem of localising diffusion sources from samples taken along *unknown* trajectories was introduced in [12] and more recently also addressed in [13]. The framework developed in [13] leverages the method in [14] and achieves estimation of the trajectory and source locations, from samples taken by a mobile sensor along piecewise linear trajectories, up to an orthogonal transformation.

In this paper, we extend the method in [13] to the case in which the trajectory is parametric, and show that also in this case we can retrieve both the source locations and the trajectory up to an orthogonal transformation. We make the assumption that the trajectory of the mobile device is defined as a linear combination of a finite number of basis functions, which are known [15]. We first consider the case in which the activation times of the sources are known and find an algebraic solution up to an orthogonal transformation. We then relax this assumption to provide a solution for the case of unknown activation times, showing how we can estimate the diffusion sources and trajectory of the mobile sensor up to a scaled orthogonal transformation.

2. PROBLEM FORMULATION

Let us consider the diffusion field induced by an instantaneous source (localized in both space and time), which will propagate in $\mathbb{R}_D$ according to the Green's function [5,6], as follows:

$$g_k(\mathbf{x},t) = \frac{a_k}{\big(4\pi\mu(t-\tau_k)\big)^{\frac{D}{2}}} e^{-\frac{||\mathbf{x}-\mathbf{S}_k||^2}{4\mu(t-\tau_k)}} H(t-\tau_k), \quad (1)$$

where:

a_k = amplitude of the diffusion source,
τ_k = activation time of the diffusion source,
$\mathbf{S}_k$ = coordinates of the source in $\mathbb{R}_D$ space,
$H(t)$ = unit step function,
μ = diffusivity of the medium.

Let us assume the field propagates in $\mathbb{R}_2$ such that $D = 2$ and denote the term $4\mu(t-\tau_k)$ with $D_k(t)$. We also assume that the observation starts at $T \geq \tau_k$, $\forall k$. We can then re-write the measurements corresponding to source k and given in Eq. (1), as follows:

$$f_k(\mathbf{x},t) = \pi g_k(\mathbf{x},t) = \frac{a_k}{D_k(t)} e^{-\frac{||\mathbf{x}-\mathbf{S}_k||^2}{D_k(t)}}.$$

Inspired by [15], suppose we take measurements along a parametric trajectory made up of L basis functions, with L unknown coefficients, of the form:

$$\mathbf{r}(t) = \sum_{j=1}^{L} \mathbf{c}_j \varphi_j(t), \quad (2)$$

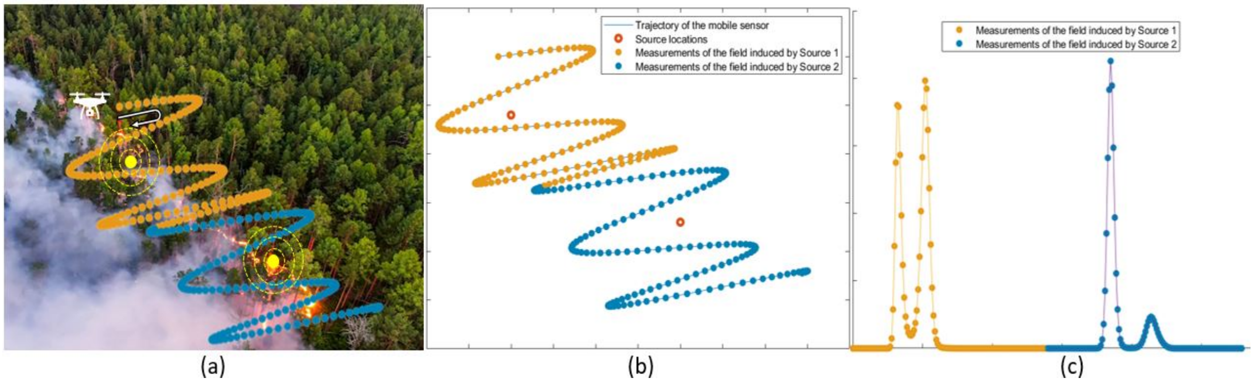

Fig. 1: The problem we consider is localising the diffusion sources and the trajectory of a mobile sensor from samples along unknown parametric trajectories, as illustrated in (a) and (b). In (c) we show the measurements taken along the trajectory in (b).

where the vectors $\mathbf{c}_j \in \mathbb{R}_2$ are multidimensional basis coefficients and the functions $\varphi_j(t)$ are known.

The cumulative measurements of the field induced by K sources is given by:

$$f(\mathbf{x}_n, t_n) = \sum_{k=1}^{K} f_k(\mathbf{x}_n, t_n) = \sum_{k=1}^{K} \frac{a_k}{D_k(t_n)} e^{-\frac{||\mathbf{x}_n - \mathbf{S}_k||^2}{D_k(t_n)}}.$$

If we assume that the diffusion sources are sufficiently separated, each measurement will only have contribution from a single source (see Fig. 1). We can then express the measurements of the field due to each source $\mathbf{S}_k$ as:

$$f_k(\mathbf{x}_n, t_n) := f_k(t_n) = \frac{a_k}{D_k(t_n)} e^{-\frac{||\mathbf{x}_n - \mathbf{S}_k||^2}{D_k(t_n)}}, \tag{3}$$

where $\mathbf{x}_n$ is the location of the measurement taken at time t_n and $\mathbf{S}_k$ is the location of the source.

Since the points $\mathbf{x}_n$ belong to the trajectory $\mathbf{r}(t)$, we can express them using Eq. (2) as:

$$\mathbf{x}_n = \sum_{j=1}^{L} \mathbf{c}_j \varphi_j(t_n), \tag{4}$$

which we can then use to get:

$$\begin{aligned}
||\mathbf{x}_n - \mathbf{S}_k||^2 &= ||\mathbf{x}_n||^2 + ||\mathbf{S}_k||^2 - 2\mathbf{S}_k^T \mathbf{x}_n \\
&= \left\|\sum_{j=1}^{L} \mathbf{c}_j \varphi_j(t_n)\right\|^2 + ||\mathbf{S}_k||^2 - 2\mathbf{S}_k^T \sum_{j=1}^{L} \mathbf{c}_j \varphi_j(t_n) \\
&= \sum_{j=1}^{L}\sum_{i=1}^{L} \varphi_j(t_n)\varphi_i(t_n)\mathbf{c}_j^T \mathbf{c}_i + ||\mathbf{S}_k||^2 - 2\sum_{j=1}^{L} \varphi_j(t_n)\mathbf{S}_k^T \mathbf{c}_j \\
&= \sum_{j=1}^{L} \varphi_j^2(t_n)||\mathbf{c}_j||^2 + 2\sum_{j=1}^{L}\sum_{i=j+1}^{L} \left(\varphi_j(t_n)\varphi_i(t_n)\mathbf{c}_j^T \mathbf{c}_i\right) \\
&+ ||\mathbf{S}_k||^2 - 2\sum_{j=1}^{L} \varphi_j(t_n)\mathbf{S}_k^T \mathbf{c}_j.
\end{aligned} \tag{5}$$

3. SIMULTANEOUS SOURCE LOCALIZATION AND TRAJECTORY RECOVERY

In this section we present a mathematical framework for simultaneous estimation of the locations of the sources and recovery of the parametric trajectory taken by a mobile sensor. We first consider the case in which the activation times of the sources are known, and then extend the method to unknown activation times.

We assume that the phenomenon we observe is induced in $\mathbb{R}_2$ by $K \geq 2$ sources where K is known, and that the trajectory is defined as in Eq. (2) using at least two independent basis functions, which are non-constant functions of time. Moreover, the diffusion sources are sufficiently separated such that each sample has contribution from a single source only. The overall number of measurements N satisfies $N \geq \frac{L(L+1)}{2} + K(L+1)$, and the number of measurements of the field of each source is assumed to be larger than $L+1$. Finally, we assume that the evolution of the diffusion field is imperceptible within the time interval of observation, such that $D_k(t) = D_k$, and that the activation times of the sources are known. For simplicity we set $D_k = 1, \forall k$. Under these assumptions, we can state the following result:

Theorem 1. *Given spatial measurements along an unknown parametric trajectory, the locations of the sources and the parameters of the trajectory can be reconstructed up to a 2D othogonal transformation, whilst the amplitudes of the sources can be retrieved exactly. If any two points in the trajectory are known, we can perfectly estimate the sources and the trajectory.*

Proof. We first show how to retrieve the inner products between the sources and the trajectory parameters. We then show how to estimate the trajectory parameters and source locations up to an orthogonal transformation.

Given the hypothesis that sources are sufficiently separated, we assume that groups of samples of the fields induced by nearby sources are separated by samples of amplitude smaller than ϵ, where $\epsilon > 0$ and $\epsilon \approx 0$[1]. Hence, we select the first $L+1$ consecutive samples with amplitude $f_1(t_n) > \epsilon$, for $n = M_1, ..., M_1 + L$, and assign them to source $\mathbf{S}_1$. We then discard the next samples, and assign the next group of $L+1$ consecutive samples satisfying $f_2(t_n) > \epsilon$, for $n = M_2, ..., M_2 + L$, to $\mathbf{S}_2$. We repeat the process for subsequent sources.

Once the measurements are paired to each source, we replace $D_k(t_n) = D_k = 1$ into Eq. (3), to obtain:

$$d_{n,k} = \ln\big(f_k(t_n)\big) = \ln(a_k) - ||\mathbf{x}_n - \mathbf{S}_k||^2.$$

We can then substitute the distances $||\mathbf{x}_n - \mathbf{S}_k||^2$ in Eq. (5), to get:

$$d_{n,k} = \ln(a_k) - ||\mathbf{S}_k||^2 - \sum_{j=1}^{L} \varphi_j^2(t_n)||\mathbf{c}_j||^2 - 2\sum_{j=1}^{L}\sum_{i=j+1}^{L} \varphi_j(t_n)\varphi_i(t_n)\mathbf{c}_j^T\mathbf{c}_i + 2\sum_{j=1}^{L}\varphi_j(t_n)\mathbf{S}_k^T\mathbf{c}_j. \quad (6)$$

Using the measurements $f_k(t_n)$, for $n = 1, 2, ..., N$, we can construct a system of N equations of the form in (6). Provided $N \geq \frac{L(L+1)}{2} + K(L+1)$, we can then solve this linear system of equations[2] to uniquely retrieve the unknowns $||\mathbf{c}_j||^2$, $\mathbf{c}_j^T\mathbf{c}_i$, $\mathbf{S}_k^T\mathbf{c}_j$ and $\ln(a_k) - ||\mathbf{S}_k||^2$, for each source $k = 1, 2, ..., K$[3] and trajectory parameters $j = 1, 2, ..., L$.

Inspired by [13], we use the estimated parameters $Y_{k,j} = \mathbf{c}_k^T\mathbf{c}_j$ to create the following matrix:

$$\mathbf{\Omega} = \begin{bmatrix} \mathbf{c}_1^T\mathbf{c}_1 & \dots & \mathbf{c}_1^T\mathbf{c}_L \\ \mathbf{c}_2^T\mathbf{c}_1 & \dots & \mathbf{c}_2^T\mathbf{c}_L \\ \vdots & \ddots & \vdots \\ \mathbf{c}_L^T\mathbf{c}_1 & \dots & \mathbf{c}_L^T\mathbf{c}_L \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{c}_1^T \\ \vdots \\ \mathbf{c}_L^T \end{bmatrix}}_{L\times 2} \underbrace{\begin{bmatrix} \mathbf{c}_1 & \dots & \mathbf{c}_L \end{bmatrix}}_{2\times L} := \mathbf{C}^T\mathbf{C}.$$

[1]For higher levels of noise, the value of ϵ should be larger in order to increase the robustness against spurious samples.

[2]In order to uniquely retrieve these unknowns, we need to impose that each of the basis functions $\varphi_j(t)$ is a non-constant function of time, i.e. $\varphi_j(t) \neq C$, for $C \in \mathbb{R}$. This system of equations may also become underdetermined for a degenerate arrangement of the sources and curve parameters, in which case it may not be possible to retrieve a unique solution.

[3]There may be cases in which the mobile sensor takes measurements in an already-visited region, such that samples $d_{n,k}$ in Eq. (6) have contribution from the same source k, for $n = N_{1,1}, ..., N_{1,2}$ and $n = N_{2,1}, ..., N_{2,2}$. Provided the system of equations has full rank, we can solve the problem also in this case. We would retrieve two sets of parameters $\mathbf{S}_{k,1}^T\mathbf{c}_j$ corresponding to the samples $n = N_{1,1}, ..., N_{1,2}$, and $\mathbf{S}_{k,2}^T\mathbf{c}_j$ corresponding to samples $n = N_{2,1}, ..., N_{2,2}$ respectively. If $\mathbf{S}_{k,1}^T\mathbf{c}_j = \mathbf{S}_{k,2}^T\mathbf{c}_j$, then these parameters would be mapped to the same source, i.e. $\mathbf{S}_{k,1} = \mathbf{S}_{k,2}$.

Given the fact that the trajectory is located in $\mathbb{R}_2$, $\mathbf{\Omega}$ is a matrix of $\mathrm{rank} \leq 2$. This means that we can factorise it using singular value decomposition as follows:

$$\mathbf{\Omega} = \mathbf{U}\mathbf{\Delta}\mathbf{V}^T,$$

where $\mathbf{U}$ is a $L \times 2$ orthogonal matrix, $\mathbf{\Delta}$ is a 2×2 diagonal matrix and $\mathbf{V}$ is a $L \times 2$ orthogonal matrix, given $L \geq 2$.

We can then obtain the estimated trajectory parameters from the SVD decomposition as follows:

$$\tilde{\mathbf{C}} = \sqrt{\mathbf{\Delta}}\mathbf{V}^T.$$

We note that if $\tilde{\mathbf{C}}$ is a solution, then $\mathbf{R}\tilde{\mathbf{C}}$ is also a solution, where $\mathbf{R}$ is an arbitrary orthogonal matrix, since $\mathbf{\Omega}$ is symmetric. As a result, the estimated curve parameters will be up to an orthogonal transformation from the true parameters.

Once the trajectory parameters $\mathbf{c}_j$ have been retrieved, the location of each source can be found from the following system of linear equations:

$$\begin{bmatrix} \mathbf{c}_1^T \\ \vdots \\ \mathbf{c}_L^T \end{bmatrix} \mathbf{S}_k = \begin{bmatrix} Y_{k,1} \\ \vdots \\ Y_{k,L} \end{bmatrix}.$$

Once the terms $||\mathbf{c}_j||^2$, $\mathbf{c}_j^T\mathbf{c}_i$ and $\mathbf{S}_k^T\mathbf{c}_j$ have been found by solving the system of linear equations described by (6), we can compute the distances between each source $\mathbf{S}_k$ and measurement $\mathbf{x}_n$ as in Eq. (5). We can then retrieve the amplitude a_k from Eq. (3), where $D_k(t) = D_k = 1$:

$$a_k = f_k(t_n)e^{||\mathbf{x}_n - \mathbf{S}_k||^2}.$$

□

The problem can also be solved in the case in which the activation time τ_k of each source k is *unknown*. In this case, the number of measurements of the field induced by each source should be larger than $1 + \frac{L(L+3)}{2}$ and hence, the total number of measurements along the trajectory must satisfy $N \geq K\left(1 + \frac{L(L+3)}{2}\right)$. Preserving the assumption that the diffusion field is not time-varying and that the sources are sufficiently separated, we can state the following result:

Theorem 2. *Given spatial measurements along an unknown parametric trajectory, the locations of the sources and the parameters of the trajectory can be reconstructed up to a scaled 2D othogonal transformation. If any two points in the trajectory are known, we can perfectly estimate the trajectory, as well as the locations, activation times and amplitudes of the sources.*

Proof. Using $1 + \frac{L(L+3)}{2}$ consecutive measurements which have contribution from source k, we can build a linear system of $1 + \frac{L(L+3)}{2}$ equations of the form in (6), from which we can retrieve the unknown parameters $\frac{||\mathbf{c}_j||^2}{D_k}$, $\frac{\mathbf{c}_j^T \mathbf{c}_i}{D_k}$, $\frac{\mathbf{S}_k^T \mathbf{c}_j}{D_k}$ and $\frac{\ln(a_k) - ||\mathbf{S}_k||^2}{D_k}$, for each source $k = 1, 2, ..., K$ and all lines $j = 1, 2, ..., L$, where $D_k = 4\mu(T - \tau_k)$.

Using $\frac{\mathbf{c}_i^T \mathbf{c}_j}{D_k}$ parameters for an arbitrary k, we can then build matrix Ω:

$$\mathbf{\Omega} = \frac{1}{D_k} \begin{bmatrix} \mathbf{c}_1^T \\ \vdots \\ \mathbf{c}_L^T \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 & \dots & \mathbf{c}_L \end{bmatrix} := \frac{1}{D_k} \mathbf{C}^T \mathbf{C}.$$

We then have that the true curve parameters can be written as $\mathbf{c}_j = \sqrt{D_k} \mathbf{R} \tilde{\mathbf{c}}_j$, where $\mathbf{R}$ is an arbitrary orthogonal transformation and $\tilde{\mathbf{c}}_j$ are the estimated trajectory parameters. This also means that any point on the true trajectory will be up to the same transformation from the corresponding point on the estimated trajectory:

$$\mathbf{x}_n = \sqrt{D_k} \mathbf{R} \tilde{\mathbf{x}}_n,$$

where x_n is defined as in Eq. (4).

Therefore, if any two points $\mathbf{x}_i$ and $\mathbf{x}_j$ are known, we can find the transformation $\sqrt{D_k}\mathbf{R}$, by solving:

$$\begin{bmatrix} \mathbf{x}_i \\ \mathbf{x}_j \end{bmatrix} = \sqrt{D_k} \mathbf{R} \begin{bmatrix} \tilde{\mathbf{x}}_i \\ \mathbf{x}_j \end{bmatrix}.$$

Given $\mathbf{R}$ is an orthogonal matrix, we can retrieve $\sqrt{D_k} = ||\mathbf{R}_1||$, where $\mathbf{R}_1$ is the first column of $\mathbf{R}$, from which $\tau_k = T - D_k$. We can then similarly estimate the activation times τ_k of all sources $k = 1, 2, ..., K$.

Finally, once the activation times have been estimated, we can find the source locations using the system of linear equations $Y_{k,j} = \frac{\mathbf{S}_k^T \mathbf{c}_j}{D_k}$, for $j = 1, 2, ..., L$ and the source amplitudes from the measurements in Eq. (3). □

Remark 1. *This method can also be extended to the case in which the sources and trajectory are located in* $\mathbb{R}_D$, *for* $D > 2$. *In this case, we would need at least* D *diffusion source and* $\frac{D(D+1)}{2}$ *independent trajectory parameters, in order to retrieve a solution up to a scaled orthogonal transformation. Finally, we would need to know the locations of* D *arbitrary points on the trajectory in order to find an exact solution.*

4. EXPERIMENTAL RESULTS

We consider a trajectory defined as in (2), where the basis functions are given by: $\varphi_1(t) = 2\cos(\frac{t}{100})$, $\varphi_2(t) = 2\sin(\frac{3t}{100})$ and $\varphi_3(t) = 0.01t + 1$.

We also assume the diffusion field is induced by two instantaneous sources, sufficiently separated in space, and with known activation times. In Fig. 2 we depict the location of the two sources together with the curvilinear trajectory of the mobile sensor. The estimation of the trajectory and sources is performed as in Section 3 and is up to an orthogonal transformation (rotation and shift) from the true trajectory and sources respectively. If any two points on the trajectory are known, the trajectory parameters and source locations are exactly retrieved.

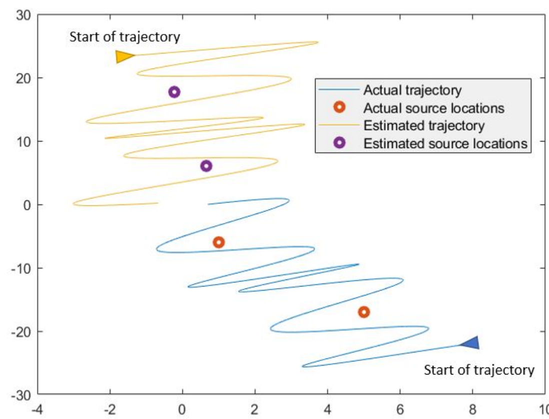

Fig. 2: True and reconstructed trajectory and source locations, in the case in which the activation times of the sources are known. The estimation is up to an orthogonal transformation.

For the scenario in Fig. 3 we consider a trajectory defined as in (2), where the basis functions are given by: $\varphi_1(t) = -0.01t^2 + 0.05t + 20$, $\varphi_2(t) = t - 20\sin(\frac{3t}{50})$ and $\varphi_3(t) = 30\cos(\frac{t}{25})$. The estimation of the trajectory and sources is exact, when any two points on the trajectory are known.

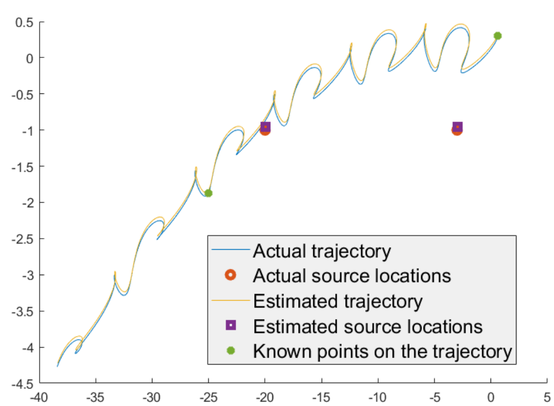

Fig. 3: Actual and reconstructed trajectory and source locations (reconstruction shifted by 0.05 for visualisation purposes), in the case in which the activation times of the sources are unknown, and we know the locations of two sample points.

Finally, within the setting of Fig. 3, we now assume that the measurements d_n are corrupted by additive white Gaussian noise, as depicted in Fig. 4 for a signal-to-noise(SNR) ratio of 20dB.

The error of the trajectory is computed as $E_{curve} = \int_0^1 \frac{||\tilde{\mathbf{r}}_j(s)-\mathbf{r}_j(s)||_2}{||\mathbf{r}_j(s)||_2} ds$, where $\tilde{\mathbf{r}}_j(s)$ and $\mathbf{r}_j(s)$ are the estimated and true trajectory respectively.

Averaged over 1000 experiments, the error in the reconstructed trajectory is $E_{curve} = 0.3848$ for SNR=10dB, $E_{curve} = 0.0077$ for SNR=20dB and $E_{curve} = 0.0065$ for SNR=30dB.

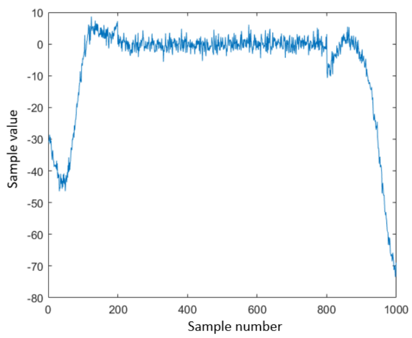

Fig. 4: Modified samples $d_{n,k}$ of the field, computed as in (6), corrupted by additive white Gaussian noise, for SNR=20dB.

5. CONCLUSIONS

In this paper we proposed a method for localising instantaneous diffusion sources, from samples taken along unknown parametric trajectories. When the activation times of the sources are known, the trajectory and source locations are retrieved up to an orthogonal transformation, whereas the source amplitudes are retrieved exactly. When the activation times of the sources are unknown, the estimation is up to a scaled orthogonal transformation. Knowing any two points in the trajectory ensures perfect estimation of the trajectory and source locations, as well as their amplitudes and activation times.

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