Approximation Order: Why the Asymptotic Constant Matters

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Abstract

We consider the approximation (either interpolation, or least-squares) of $\mathbf{L}^2$ functions in the shift-invariant space $\mathcal{V}_T = \mathrm{span}_{k\in\mathbb{Z}}\{\varphi(\frac{t}{T} - n)\}$ that is generated by the single shifted function φ. We measure the approximation error in an $\mathbf{L}^2$ sense and evaluate the asymptotic equivalent of this error as the sampling step T tends to zero. Let $f \in \mathbf{L}^2$ and f_T be its approximation in $\mathcal{V}_T$. It is well-known that, if φ satisfies the Strang-Fix conditions of order L, and under mild technical constraints, $\|f - f_T\|_{\mathbf{L}^2} = O(T^L)$ [4].

In this presentation however, we want to be more accurate and concentrate on the constant C_φ which is such that

$$\|f - f_T\|_{\mathbf{L}^2} = C_\varphi \|f^{(L)}\|_{\mathbf{L}^2} T^L + o(T^L).$$

We showed previously how to compute this constant [2, 3, 5, 6]. We showed that the numerical values associated to specific, widely-used kernels φ exhibit substantial variations. This important observation motivates our presentation, because the asymptotic approximation constant is a very good indicator of performance. Letting φ_1 and φ_2 be two generators of order L, we define the "sampling gain" of φ_1 over φ_2 by

$$\gamma_{\varphi_1/\varphi_2} = \left(\frac{C_{\varphi_1}}{C_{\varphi_2}}\right)^{-\frac{1}{L}}.$$

This quantity is interpreted as the factor by which the approximation using φ_2 has to be over/down-sampled in order to exhibit the same asymptotic

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error as that using φ_1. For instance, we will prove that, when the approximation order tends to ∞, Daubechies' scaling functions require π-times more coefficients than the same-order spline-approximation, asymptotically [3].

Given an approximation order L, we will also give explicit expressions of the smallest-support kernels whose approximation constant is minimal. These functions are called "OMOMS" [1]. We will see that our new kernels bring a huge gain over splines of same order, and, typically, that this gain increases linearly as the order increases: $\gamma_{\text{OMOMS/spline}} \approx \frac{2}{\pi e}L$.

Finally, we will shift the kernel $\varphi(t) \rightsquigarrow \varphi_\tau(t) = \varphi(t-\tau)$ which yields a new interpolation space that has the same least-squares approximation constant $C^{\text{LS}}_{\varphi_\tau} = C^{\text{LS}}_{\varphi}$, but a different interpolation constant $C^{\text{I}}_{\varphi_\tau} \geq C^{\text{LS}}_{\varphi_\tau}$. We will prove that it is always possible to choose τ so that $C^{\text{I}}_{\varphi_\tau} = \min_{\tau'} C^{\text{I}}_{\varphi_{\tau'}} = C^{\text{LS}}_{\varphi}$. For example, we will see that, for the linear spline, one has $\tau = \frac{1}{2}(1-\frac{1}{\sqrt{3}}) \approx 0.21$, and that this optimal value gets closer to $\frac{1}{4}$ as higher-order splines of odd degree are considered.

All our theoretical claims will be substantiated with computer experiments, some of which are already available as Java demos on our web site [7].

References

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