A Quantitative Fourier Analysis of the Linear Approximation Error by Wavelets

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Introduction We introduce a simple method —integration of the power spectrum against a Fourier kernel— for computing the approximation error by wavelets. This method is powerful enough to recover all classical $\mathbf{L}^2$ results in approximation theory (Strang-Fix theory [3]), and also to provide new error estimates that are sharper and asymptotically exact.

Approximation theoretical results Given an $\mathbf{L}^2$ function $f(x)$, we compute its approximation $\mathcal{Q}_T f$ over the T-integer shift invariant space $\mathrm{span}_{n\in\mathbb{Z}}\left\{\varphi(\frac{x}{T}-n)\right\}$ by $\mathcal{Q}_T f(x) = \langle f(T\xi), \tilde\varphi(\xi-n)\rangle\, \varphi(\frac{x}{T}-n)$, where $\varphi, \tilde\varphi$ are assumed to satisfy some mild conditions.

Our main theorem provides an evaluation of the $\mathbf{L}^2$ approximation error $\|f-\mathcal{Q}_T f\|_{\mathbf{L}^2}$. If we assume that $|\omega|^r \hat f(\omega)$ is in $\mathbf{L}^2$, then

$$\|f-\mathcal{Q}_T f\|_{\mathbf{L}^2} = \underbrace{\sqrt{\int |\hat f(\omega)|^2 E(\omega T)\,\frac{d\omega}{2\pi}}}_{\eta_T(f)} + O(T^r) \tag{1}$$

with $E(\omega) = 1 - 2\Re\{\overline{\hat{\tilde\varphi}(\omega)}\hat\varphi(\omega)\} + |\hat{\tilde\varphi}(\omega)|^2 \sum_n |\hat\varphi(\omega+2n\pi)|^2$. The $O(T^r)$ term can be bounded very accurately [1]; moreover, it cancels when f is bandlimited over $[-\frac{1}{2T}, \frac{1}{2T}]$. An application of (1) is a new proof of the Strang-Fix equivalence [3], under general conditions, and the extension of this equivalence to multi-wavelets [1].

The second essential result states that the average of the approximation error over delays of f is exactly $\eta_T(f)^2$. Thus, the $O(T^r)$ in (1) cancels on the average which shows that this estimate is non-biased.

Quantitative results for dyadic wavelets Using (1), asymptotic expansions of the approximation error as $T\to 0$ can be obtained. When φ satisfies a two-scale equation, it is even possible to express the coefficients of the expansion only in function of the generating filter [2]. As an example we provide the first terms of the developments of Daubechies' approximation error

$$\|f-\mathcal{Q}_T f\|^2_{\mathbf{L}^2} = \underbrace{\frac{2^{-4L-1}\binom{2L}{L}}{1-2^{-2L}}}_{D_L} \|s^{(L)}\|^2_{\mathbf{L}^2} T^{2L} - \frac{2^{-4L}(2L-1)\binom{2L}{L+1}}{3(1-2^{-2L-2})}\|s^{(L+1)}\|^2_{\mathbf{L}^2} T^{2L+2} + O(T^{2L+4}) \tag{2}$$

Using the exact expressions for the first order constants of splines and Daubechies, we prove that, asymptotically, Daubechies' wavelets require a sampling grid that is π-*times tighter* than splines of the same order to achieve the same accuracy.

Upper bounds can also be derived in the same manner, some of them being asymptotically sharp. For instance, we obtain the following asymptotically exact upper bound for the least-squares spline approximation of degree $L-1$

$$\|f-\mathcal{Q}_T f\|_{\mathbf{L}^2} \le \underbrace{\frac{\sqrt{2\zeta(2L)}}{(2\pi)^L}}_{S_L} T^L \|f^{(L)}\|_{\mathbf{L}^2} + \frac{\sqrt{2}}{\pi^{L+1}} T^{L+1}\|f^{(L+1)}\|_{\mathbf{L}^2} \tag{3}$$

and the following first order bound for Daubechies' least squares approximation

$$\|s-\mathcal{Q}_T s\|_{\mathbf{L}^2} \le D_L \sqrt{1+\frac{1}{2}\left[\frac{S_L}{D_L}\right]^2}\, \|s^{(L)}\|_{\mathbf{L}^2} T^L \tag{4}$$

Note that, for L large enough, $S_L \ll D_L$ which shows the sharpness of this bound.

References

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[3] G. Strang and G. Fix, *A Fourier analysis of the finite element variational method*, in Constructive Aspect of Functional Analysis, Cremonese, ed., Rome, 1971, pp. 796–830.