Short Basis Functions for Constant-Variance Interpolation

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ABSTRACT

An interpolation model is a necessary ingredient of intensity-based registration methods. The properties of such a model depend entirely on its basis function, which has been traditionally characterized by features such as its order of approximation and its support. However, as has been recently shown, these features are blind to the amount of registration bias created by the interpolation process alone; an additional requirement that has been named *constant-variance* interpolation is needed to remove this bias.

In this paper, we present a theoretical investigation of the role of the interpolation basis in a registration context. Contrarily to published analyses, ours is deterministic; it nevertheless leads to the same conclusion, which is that constant-variance interpolation is beneficial to image registration.

In addition, we propose a novel family of interpolation bases that can have any desired order of approximation while maintaining the constant-variance property. Our family includes every constant-variance basis we know of. It is described by an explicit formula that contains two free functional terms: an arbitrary 1-periodic binary function that takes values from $\{-1, 1\}$, and another arbitrary function that must satisfy the partition of unity. These degrees of freedom can be harnessed to build many family members for a given order of approximation and a fixed support. We provide the example of a symmetric basis with two orders of approximation that is supported over $[-\frac{3}{2}, \frac{3}{2}]$; this support is one unit shorter than a basis of identical order that had been previously published.

Keywords: Multiresolution and wavelets, Registration.

1. INTRODUCTION

Image registration, which is the process of aligning a source image to a target image, is a necessary ingredient of many medical procedures. Typical applications are detection/analysis of change and atlas-based segmentation. It generally proceeds by applying a tunable geometric transformation to the source image, and by adjusting the tuning parameters so that the transformed source becomes as close as possible to the target. This closeness is measured by some objective criterion that allows for an automated procedure.

As data are available through their samples only, and as a geometric transformation is involved, an interpolation model is always required to transform the source image. In many cases, this interpolation model is explicit. But even when it is not, for example when the registration criterion is the measure of mutual information computed thanks to the so-called partial-volume method of Maes *et al.*[1] or its generalization by Chen *et al.*,[2] the underlying implicit interpolation model can nevertheless be retrieved—in these particular cases, it can be shown to be no better than nearest-neighbor, as hinted by the fact that the construction of the joint histogram involves no intermediate values beyond those found in the data samples.

On one hand, it has been long known that all interpolation models have not been created equal. They differ by their computational requirements, and by their ability to faithfully reproduce the continuous function from which the data samples have been obtained. An extended literature is concerned with analyzing those aspects in a general imaging context, such as the work of Lehman *et al.*[3,4] On the other hand, researchers like Rohde *et al.*[5,6] and Salvado *et al.*[7] have recently started to tailor their interpolation model to the specific problem of image registration. In this context, the main observation is that traditional interpolation models do not maintain the variance of the transformed source image while it is translated, even though this would be highly desirable

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because many of the traditional photometric (*i.e.*, intensity-based, or landmark-ignorant) registration criteria rely at least in part on this variance. Not only does a translation-dependent variance obviously create a registration bias when the objective registration criterion is the sum of squared differences between image samples, it also creates one for other criteria such as normalized cross-correlation, or even for mutual information, as has been shown by Rohde *et al.*[6]

In this paper, we propose a family of basis functions that ensure the conservation of variance. Our new bases are symmetric and interpolating, but are allowed to be discontinuous. They are characterized by their order of approximation L and are supported over $[-(L+1)/2, (L+1)/2]$. As they are not made of polynomial pieces, they fall outside the scope of an earlier paper by Blu *et al.*[8] At identical order of approximation, our basis function is shorter than the basis proposed by Salvado *et al.*[7]

2. CONSTANT-VARIANCE INTERPOLATION

2.1. Interpolation

Interpolation is the process of building a continuously defined function f out of a sequence of samples s. In q dimensions, we can formally describe this process as a transformation $\mathcal{T}$ so that $\forall \mathbf{x} \in \mathbb{R}^q : f(\mathbf{x}) = \mathcal{T}\{s[\cdot]\}(\mathbf{x})$. Except in specialized cases that have to do with perceptual issues, this transformation is oftentimes chosen to be linear, in the sense that $\forall \lambda \in \mathbb{C} : \mathcal{T}\{\lambda\, s_1[\cdot] + s_2[\cdot]\}(\mathbf{x}) = \lambda\, f_1(\mathbf{x}) + f_2(\mathbf{x})$. It is also chosen to be integer-shift invariant, so that $\forall \mathbf{k} \in \mathbb{Z}^q : \mathcal{T}\{s[\cdot - \mathbf{k}]\}(\mathbf{x}) = f(\mathbf{x} - \mathbf{k})$. Altogether, a generalized form that satisfies these requirements is

$$\forall \mathbf{x} \in \mathbb{R}^q : f(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^q} c[\mathbf{k}]\, \varphi(\mathbf{x} - \mathbf{k}), \tag{1}$$

where the sequence of coefficients c is related to the sequence of samples s in such a way that $\forall \mathbf{k}_0 \in \mathbb{Z}^q : \sum_{\mathbf{k}\in\mathbb{Z}^q} c[\mathbf{k}]\, \varphi(\mathbf{k}_0 - \mathbf{k}) = s[\mathbf{k}_0]$, which ensures the desired interpolation condition $\forall \mathbf{k}_0 \in \mathbb{Z}^q : f(\mathbf{x})|_{\mathbf{x}=\mathbf{k}_0} = s[\mathbf{k}_0]$.

Clearly, some technical conditions are required i) to ensure convergence of the sum in (1), and ii) to ensure that the relation between c and s is one-to-one. Within the scope of this paper we shall drastically overconstrain requirements i) and ii) by restricting our attention to bounded and finite-support bases φ (this automatically ensures convergence when $c \in \ell_1$), and by imposing that $c = s$. This last choice transforms the interpolation condition into $\forall \mathbf{k}_0 \in \mathbb{Z}^q : \sum_{\mathbf{k}\in\mathbb{Z}^q} s[\mathbf{k}]\, \varphi(\mathbf{k}_0 - \mathbf{k}) = s[\mathbf{k}_0]$; selecting the unit impulse—a.k.a. Kronecker delta—in the role of the arbitrary sequence s results in $\forall \mathbf{k}_0 \in \mathbb{Z}^q : \varphi(\mathbf{k}_0) = \delta[\mathbf{k}_0]$. Conversely, if we first impose that $\forall \mathbf{k} \in \mathbb{Z}^q : \varphi(\mathbf{x})|_{\mathbf{x}=\mathbf{k}} = \delta[\mathbf{k}]$, then it follows from (1) that $\forall \mathbf{k} \in \mathbb{Z}^q : f(\mathbf{x})|_{\mathbf{x}=\mathbf{k}} = s[\mathbf{k}]$. In other words, setting $c = s$ is *equivalent* to restricting the general basis φ to be an interpolating basis ϕ that satisfies $\forall \mathbf{k} \in \mathbb{Z} : \phi(\mathbf{k}) = \delta[\mathbf{k}]$. We then rewrite (1) as

$$\forall \mathbf{x} \in \mathbb{R}^q : f(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^q} s[\mathbf{k}]\, \phi(\mathbf{x} - \mathbf{k}). \tag{2}$$

2.2. Variance

Suppose we want to register the warped version of a source image s to a target image t, and suppose further that the objective criterion $J\{s, t\}$ is the sum of square differences (SSD, *i.e.*, the ℓ_2 norm) between t and the geometrically transformed version $\Gamma\{s\}$ of s obtained by interpolating s as in (2), so that $\forall \mathbf{x} \in \mathbb{R}^q : \Gamma\{s, \mathbf{p}\}(\mathbf{x}) = f(\mathbf{g}(\mathbf{x}, \mathbf{p}))$, where $\mathbf{g}$ is the geometric transformation parameterized by $\mathbf{p}$. Then, we write that $J\{s, t\}(\mathbf{p}) = \|\Gamma\{s, \mathbf{p}\} - t\|^2_{\ell_2} = \|\Gamma\{s, \mathbf{p}\}\|^2_{\ell_2} + \|t\|^2_{\ell_2} - 2\, \langle\Gamma\{s, \mathbf{p}\}, t\rangle_{\ell_2}$. In the right-hand side, the last term (the discrete cross-correlation between $\Gamma\{s\}$ and t) drives the registration and carries the relevant information; the middle term (the discrete squared norm of t) is constant and does not participate to the minimization of J in terms of $\mathbf{p}$. But the first term is troublesome: because $\|\Gamma\{s, \mathbf{p}\}\|^2_{\ell_2}$ formally depends on s and $\mathbf{p}$ alone, and not on t, this quantity can bias the registration if the dependence on $\mathbf{p}$ is effective.

From (2) we deduce that

$$\|\Gamma\{s, \mathbf{p}\}\|^2_{\ell_2} = \sum_{\mathbf{k}_1\in\mathbb{Z}^q} \sum_{\mathbf{k}_2\in\mathbb{Z}^q} s[\mathbf{k}_1]\, s[\mathbf{k}_2] \underbrace{\sum_{\mathbf{k}\in\mathbb{Z}^q} \phi(\mathbf{g}(\mathbf{k}, \mathbf{p}) - \mathbf{k}_1)\, \phi(\mathbf{g}(\mathbf{k}, \mathbf{p}) - \mathbf{k}_2)}_{\Phi_{\mathbf{k}_1,\mathbf{k}_2}(\mathbf{p})},$$

which is easier to analyze if we focus on translations described by $\mathbf{g}(\mathbf{x},\Delta\mathbf{x})=\mathbf{x}+\Delta\mathbf{x}$. Then, $\Phi_{\mathbf{k}_1,\mathbf{k}_2}(\Delta\mathbf{x})=\sum_{\mathbf{k}\in\mathbb{Z}^q}\phi(\mathbf{k}+\Delta\mathbf{x}-\mathbf{k}_1)\,\phi(\mathbf{k}+\Delta\mathbf{x}-\mathbf{k}_2)=\sum_{\mathbf{k}\in\mathbb{Z}^q}\phi(\Delta\mathbf{x}+\mathbf{k})\,\phi(\Delta\mathbf{x}+\mathbf{k}+\Delta\mathbf{k})=\Phi_{\Delta\mathbf{k}}(\Delta\mathbf{x})$, with $\Delta\mathbf{k}=\mathbf{k}_1-\mathbf{k}_2$. Clearly, $\Phi_{\Delta\mathbf{k}}(\Delta\mathbf{x})$ is $\mathbf{1}$-periodic in $\Delta\mathbf{x}$ and reaches the maximum value $\Phi_{\mathbf{0}}(\Delta\mathbf{x})=\sum_{\mathbf{k}\in\mathbb{Z}^q}\left(\phi(\Delta\mathbf{x}+\mathbf{k})\right)^2$. As s is arbitrary, we can best ensure the desirable property $\|\Gamma\{s,\Delta\mathbf{x}\}\|_{\ell_2}^2=\|s\|_{\ell_2}^2$ by choosing ϕ so that $\Phi_{\Delta\mathbf{k}}(\Delta\mathbf{x})=\delta[\Delta\mathbf{k}]$. It is unfortunately not always possible to satisfy this requirement exactly for all $\Delta\mathbf{k}\in\mathbb{Z}^q$ and for all $\Delta\mathbf{x}\in\mathbb{R}^q$; we therefore concentrate on the dominating term and apply ourselves to constructing basis functions ϕ so that

$$\Phi_{\mathbf{0}}(\Delta\mathbf{x})=1. \tag{3}$$

While we have performed this analysis in the restricting case of a translation and of a SSD criterion, we insist that the outcome has a broader scope. First, any general geometric transformation $\mathbf{g}$ can be understood both as a warping function or as a displacement map. The latter interpretation highlights local translation as paramount. Second, Rohde *et al.*[5] have shown that a non-constant term $\Phi_{\mathbf{0}}$ creates a registration bias not only for SSD, but also for normalized correlation and mutual information too. Contrarily to our present analysis, which considers only deterministic signals, theirs includes stochastic elements as well. Both approaches point to the importance of (3) in the context of image registration.

3. STATE OF THE ART

To the best of our knowledge, the collection of published basis functions that enforce (3) is limited. A trivial one is the rect function defined by $\forall x\in\mathbb{R}:\mathrm{rect}(x)=\frac{1}{2}\left(\mathrm{sign}(x+\frac{1}{2})-\mathrm{sign}(x-\frac{1}{2})\right)$. Since this function has unit support, it is trivial to see that $\forall x\in\mathbb{R}\setminus\frac{1}{2}\mathbb{Z}:\sum_{k\in\mathbb{Z}}\left(\mathrm{rect}(x+k)\right)^2=1$. Even though it also happens that $\forall\Delta k\in\mathbb{Z}\setminus\{0\}:\sum_{k\in\mathbb{Z}}\mathrm{rect}(x+k)\,\mathrm{rect}(x+\Delta k+k)=0$, the rect function is not favored because of its low order of approximation. Another function that satisfies (3) is $\forall x\in\mathbb{R}:\mathrm{sinc}(x)=\frac{\sin(\pi\,x)}{\pi\,x}$. Using Poisson, we have that $\forall x\in\mathbb{R}:\sum_{k\in\mathbb{Z}}\left(\mathrm{sinc}(x+k)\right)^2=\sum_{n\in\mathbb{Z}}\mathcal{F}\{(\mathrm{sinc}(x+\cdot))^2\}(2\,\pi\,n)=\sum_{n\in\mathbb{Z}}e^{j\,2\,\pi\,n\,x}\,\mathrm{tri}(n)=1$, where tri is defined by $\forall x\in\mathbb{R}:\mathrm{tri}(x)=\max(0,1-|x|)$. More generally, Poisson can also be used to show that $\forall\Delta k\in\mathbb{Z}:\Phi_{\Delta k}=\delta[\Delta k]$ when $\phi=\mathrm{sinc}$. But like rect, the sinc basis is not favored, this time because of its slow spatial decay. Truncation or more gradual apodization using standard windows is not recommended either because the order of approximation would collapse, and (3) would also cease to hold true.

A finite-support basis function that satisfies (3) has been recently developed by Salvado *et al.*[7] While these authors have originally presented it by the way of a computational procedure, it is more appropriate, for analysis purposes, to realize that following their procedure is equivalent to using an explicit equivalent basis φ_S. We determine this basis to be given by

$$\varphi_S(x)=\begin{cases}\frac{26\,x^2-10+(2-3\,|x|)\left(2\,x^2+8+3\,\sqrt{-4\,|x|\,(1-|x|)\,(|x|\,(1-|x|)+5\,\lambda-2)+6\,\lambda-2}\right)}{18-60\,|x|\,(1-|x|)} & |x|\leq 1\\ \frac{-(2-|x|)\left(6\,|x|\,(3-|x|)-14+\sqrt{-4\,|x|\,(3-|x|)\,(|x|\,(3-|x|)+5\,\lambda-6)+46\,\lambda-34}\right)}{46-20\,|x|\,(3-|x|)} & 1<|x|\leq 2\\ 0 & 2<|x|,\end{cases} \tag{4}$$

where $\lambda>\frac{1}{3}$ is a tuning parameter. This basis is symmetric; its support is $[-2,2]$, and it has two orders of approximation—the same approximation order as tri. It is continuous but not continuously differentiable in general, except when $\lambda=1$, the only parameter value for which $\varphi_S=\phi_S$ is interpolant. The computational procedure proposed by Salvado *et al.*[7] suggests that this basis be used as a quasi-interpolator when $\lambda\neq 1$; this has the drawback that the data samples are not always represented faithfully. We show some φ_S members in Figure 1.

4. A NEW FAMILY OF BASIS FUNCTIONS

4.1. Properties

In this paper, we propose to build a novel family of basis functions that satisfy (3) and that includes ϕ_S. Our goal is to index the new bases according to their order of approximation L, which has been shown by Thévenaz *et al.*,[9] among others, to be the main quality measure of any basis function. By default, we ask that at least

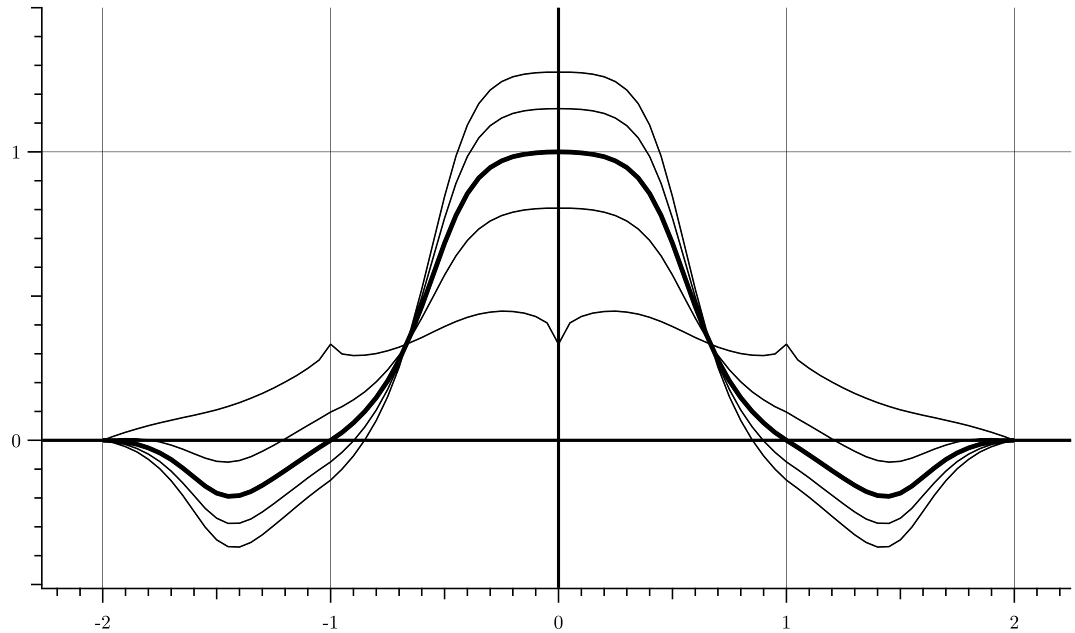

Figure 1. Family of basis functions that underly the computational procedure proposed by Salvado *et al.*[7] The most irregular curve corresponds to $\lambda = \frac{1}{3}$. The thick line corresponds to $\lambda = 1$, the only case for which ϕ_S is interpolant and continuously differentiable.

one order of approximation be present, which is equivalent to requesting that the partition of unity be satisfied so that

$$\forall x \in \mathbb{R} : \sum_{k \in \mathbb{Z}} \varphi(x - k) = 1. \tag{5}$$

This ensures that flat image areas can be well represented. Additional orders of approximation are obtained whenever (5) is satisfied jointly with the remaining Strang-Fix conditions given by

$$\forall n \in [1 \ldots L - 1], \forall x \in \mathbb{R} : \sum_{k \in \mathbb{Z}} (x - k)^n \ \varphi(x - k) = \mu_n, \tag{6}$$

where μ_n is a number that depends on n only, not on x, and where L is the overall number of orders of approximation (which includes (5)). To collect every condition in just one place, we finally rewrite here (3) as

$$\forall x \in \mathbb{R} : \sum_{k \in \mathbb{Z}} (\varphi(x - k))^2 = 1. \tag{7}$$

By setting $x = 0$ in (7), we conclude that φ cannot vanish for every integer. Therefore, there exists at least one $k_0 \in \mathbb{Z}$ such that $\varphi(k_0) \neq 0$. Because of the integer-shift property of interpolation, without loss of generality we can assume that $k_0 = 0$; consequently, for normalization purposes, we can safely impose that $\varphi(0) = 1$. But then, it can be concluded from (7) with $x = 0$ that $\varphi(k) = 0$ for all remaining terms indexed by $k \in \mathbb{Z} \setminus \{0\}$; therefore, $\varphi = \phi$ is necessarily interpolating if it is to satisfy both (7) and $\varphi(0) = 1$. In this case, and by setting $x = 0$ in (6), we have that $\sum_{k \in \mathbb{Z}} (-k)^n \ \phi(-k) = \mu_n$; as ϕ is interpolating, we conclude that μ_n must necessarily vanish for $n \in [1 \ldots L - 1]$.

4.2. General Construction

With $L \in \mathbb{N} \setminus \{0\}$, consider now the following construction:

$$\forall x \in \mathbb{R} : \psi(x) = \beta_{\mathrm{I}}^{L-1}(x) + \theta(x) \, \Delta^L \{\varphi_0\}(x), \tag{8}$$

where $\beta_{\rm I}^{L-1}$ is the interpolating I-MOMS of degree $(L-1)$ defined in an earlier work by Blu *et al.*,[10] where $\theta(x)=\theta(x+1)$ is a 1-periodic function to be examined shortly, where Δ^L is the central finite-difference operator of order L defined as $\Delta^L\{f\}(x)=\sum_{n=0}^{L}\,(-1)^n\,\binom{L}{n}\,f(x+\frac{L}{2}-n)$, and where φ_0 is any function that satisfies (5). Let us first check that ψ satisfies (5) by writing that $\mu_0(x)=\sum_{k\in\mathbb{Z}}\,\psi(x-k)=\sum_{k\in\mathbb{Z}}\,\beta_{\rm I}^{L-1}(x-k)+\sum_{k\in\mathbb{Z}}\,\theta(x-k)\,\Delta^L\{\varphi_0\}(x-k)$. Because $\beta_{\rm I}^{L-1}$ has order L, because θ is 1-periodic, and because the $\sum$ and Δ operators commute here, we have that $\mu_0(x)=1+\theta(x)\,\Delta^L\{\sum_{k\in\mathbb{Z}}\,\varphi_0(\cdot-k)\}(x)$. Finally, we see that $\mu_0(x)=1$ since φ_0 itself satisfies (5). For similar reasons and because $\beta_{\rm I}^{L-1}$ is interpolating, we see that (8) also satisfies (6) since, for $0<n<L$, we have that $\mu_n(x)=\sum_{k\in\mathbb{Z}}\,(x-k)^n\,\psi(x-k)=\sum_{k\in\mathbb{Z}}\,(x-k)^n\,\beta_{\rm I}^{L-1}(x-k)+\sum_{k\in\mathbb{Z}}\,(x-k)^n\,\theta(x-k)\,\Delta^L\{\varphi_0\}(x-k)=\theta(x)\,(-1)^n\,\sum_{k\in\mathbb{Z}}\,\varphi_0(x+\frac{L}{2}-k)\,\Delta^L\{\left(\cdot-\frac{L}{2}\right)^n\}(k-x)=0$, where we have used the fact that finite differences of order L annihilate polynomials of lesser degree. From $\mu_0(x)=1$ and $\mu_n(x)=0$, we conclude that ψ is interpolating and has L orders of approximation.

Next, we have to determine θ so that (7) is satisfied as well. To that effect, we have to solve for $1=\sum_{k\in\mathbb{Z}}\,(\psi(x-k))^2$ in terms of θ. Using the assumption that θ is 1-periodic leads to a quadratic equation with solution

$$\begin{aligned}\forall x\in\mathbb{R}:\theta(x)=\frac{1}{\sum_{k\in\mathbb{Z}}\,\left(\Delta^L\{\varphi_0\}(x-k)\right)^2}\,\Bigg(&-\sum_{k\in\mathbb{Z}}\beta_{\rm I}^{L-1}(x-k)\,\Delta^L\{\varphi_0\}(x-k)\\&+\xi(x)\sqrt{\left(\sum_{k\in\mathbb{Z}}\beta_{\rm I}^{L-1}(x-k)\,\Delta^L\{\varphi_0\}(x-k)\right)^2+\left(1-\sum_{k\in\mathbb{Z}}\left(\beta_{\rm I}^{L-1}(x-k)\right)^2\right)\left(\sum_{k\in\mathbb{Z}}\left(\Delta^L\{\varphi_0\}(x-k)\right)^2\right)}\Bigg),\end{aligned}\tag{9}$$

where ξ is a 1-periodic function that takes values from the set $\{-1,1\}$ and that satisfies $\xi(0)=\mathrm{sign}(\Delta^L\{\varphi_0\}(0))$, so that $\theta(0)=0$, which in turn enforces $\psi(0)=1$. Except for this pointwise condition, the 1-periodic bi-valued function ξ is arbitrary—provided it satisfies some mildly restrictive additional technicalities. Finally, it is easy to check that θ turns out to be 1-periodic too, as required in the derivation above. Moreover, it turns out that the argument of the square root in (9) is guaranteed to be nonnegative (analysis not shown).

4.3. Specific Examples

We can take advantage of the freedom offered by the choice of ξ and φ_0 to build many different functions ψ for a given order of approximation. For instance, the description of φ_S can be compressed into just $L=2$ and $\varphi_0=\mathrm{tri}$ with $\xi=-1$, which is much more compact than the tiresome description given in (4). But tri is not the shortest-support function φ_0 that satisfies (5); the function rect is even shorter. To reduce computational costs without sacrificing quality, we therefore propose to take advantage of this fact.

We show in Figures 2–4 three widely different outcomes. In the first case, we have set $L=2$, $\varphi_0=\mathrm{rect}$, and $\xi(x)=\mathrm{sign}(\Delta^2\{\mathrm{rect}\}(0))=-1$. This generates a well-behaved ψ that is symmetric and not too irregular—please note that additional orders of regularity come at the cost of an increase of the support, which is less affordable than losing regularity. In the second case, we have set $L=2$, $\varphi_0=\mathrm{rect}$, and $\xi(x)=\mathrm{sign}(\frac{1}{2}-\mathrm{sign}(\cos\frac{\pi}{x+\frac{1}{4}}))$ for $x\in[-\frac{1}{2},\frac{1}{2}[$, where the duplicate use of the sign functions ensures that ξ does not untowardly vanish; then, we have built ξ by periodizing this arbitrary function. The resulting ψ is highly irregular, with an infinite number of zero-crossings. This is an example of one of those rare cases for which the midly restrictive technicalities we were mentioning are not satisfied, since (5) does not converge for $x=\frac{3}{4}$. Finally, we show in Figure 4 a basis function that possesses the higher order $L=3$ obtained with $\varphi_0=\mathrm{rect}$, and $\xi(x)=\mathrm{sign}(\frac{1}{2}-x+\lfloor x\rfloor)$.

Examining the support of our new construction ψ, we observe that it is dominated by $\left(\mathrm{support}\{\Delta^L\{\mathrm{rect}\}\}\right)=[-\frac{L+1}{2},\frac{L+1}{2}]$, since $\left(\mathrm{support}\{\beta_{\rm I}^{L-1}\}\right)=[-\frac{L}{2},\frac{L}{2}]$ is shorter. As the order of approximation of ϕ_S happens to be $L_S=2$, it turns out that the family of functions ψ that we just derived contains members (*e.g.*, see Figure 2) that possess the same order $L=2$ but that have a shorter support than that of ϕ_S; in addition, our family also contains members (*e.g.*, see Figure 4) of identical support $[-2,2]$ but that possess the higher order of approximation $L=3$.

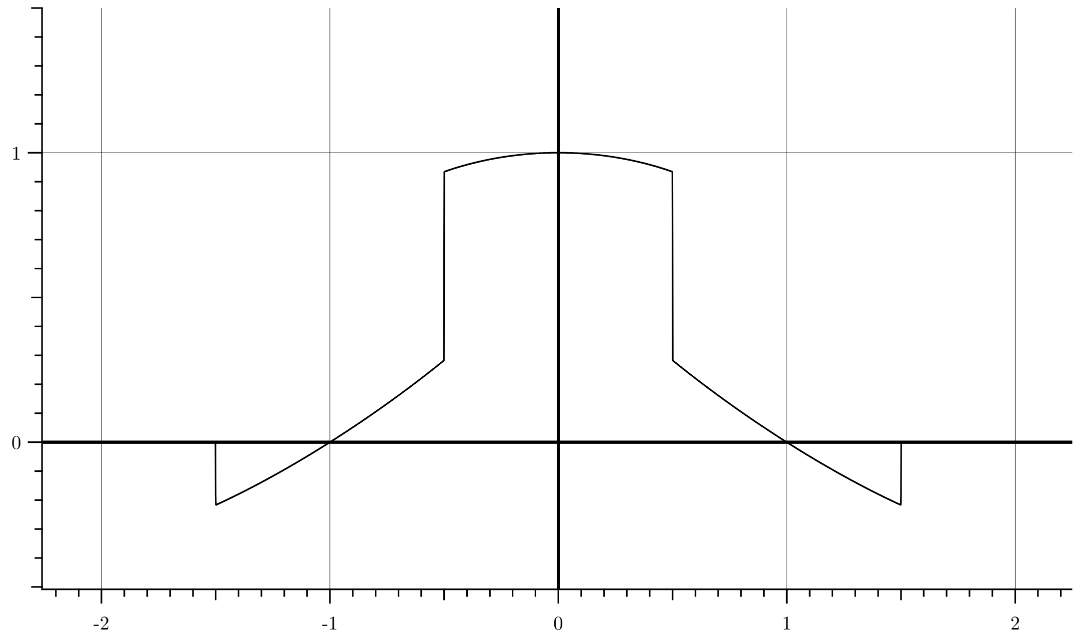

Figure 2. Proposed new basis function with order of approximation $L = 2$. This order is identical to that of φ_S, but our proposed basis is shorter; yet, it has a larger capture range.

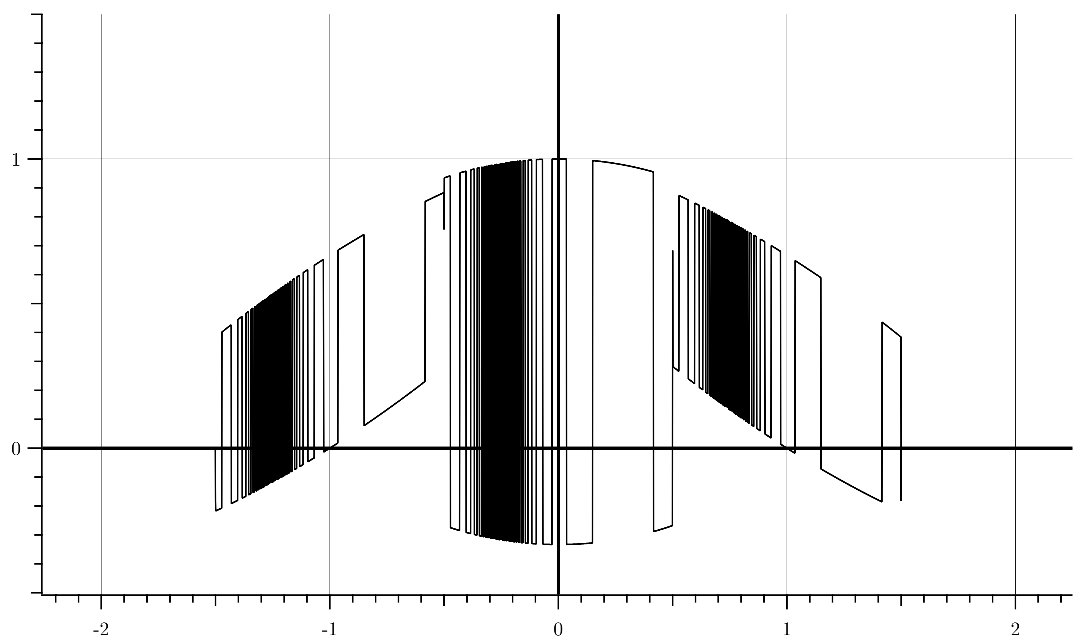

Figure 3. Another basis function supported over $[-\frac{3}{2}, \frac{3}{2}]$, with order of approximation $L = 2$. This pathological basis has an infinite number of zero-crossings; it satisfies the partition of unity almost everywhere, but does not always offer pointwise convergence.

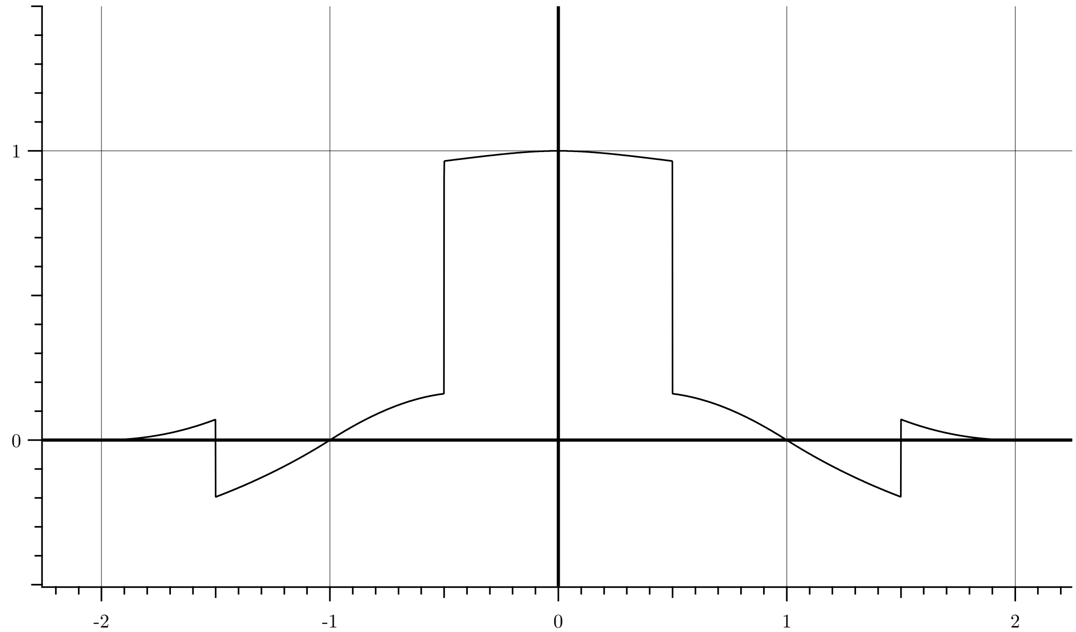

Figure 4. A basis function supported over $[-2, 2]$ with the order of approximation $L = 3$, which is one higher than that for ϕ_S in Figure 1.

4.4. Stochastic Analysis

Let us now abandon deterministic signals in favor of stochastic ones. To do that, we shall admit that the sample $f_n[\mathbf{k}]$ at location $\mathbf{k} \in \mathbb{Z}^q$ is the n-th realization of a random variable $F[\mathbf{k}]$ with expected value $\mathrm{E}\{F[\mathbf{k}]\} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_n[\mathbf{k}]$. Moreover, we make the hypothesis that all samples are identically distributed as far as their expected value and their autocorrelation are concerned, so that $\forall \mathbf{k}_1, \mathbf{k}_2 \in \mathbb{Z}^q : \mathrm{E}\{F[\mathbf{k}_1]\} = \mathrm{E}\{F[\mathbf{k}_2]\} = \bar{f}$ and so that $\mathrm{E}\{F[\mathbf{k}_1]\, F[\mathbf{k}_1]\} = C$. Finally, we say that the samples are uncorrelated, which amounts to $\mathrm{E}\{F[\mathbf{k}_1]\, F[\mathbf{k}_2]\} = C\, \delta[\mathbf{k}_1 - \mathbf{k}_2]$. These assumptions typically describe the noise component of a signal; therefore, we refer to the present analysis as to the pure-noise case. If we want to determine an expected contribution to $J\{f, f\}(\Delta\mathbf{x})$ at location $\mathbf{k}_0$, then we have to compute $J_0 = \mathrm{E}\{(F(\mathbf{k}_0) - F(\mathbf{k}_0 + \Delta\mathbf{x}))^2\}$. To do so, we first use the equal-mean assumption and (5) to compute $\forall \mathbf{x} \in \mathbf{Z}^q : \mathrm{E}\{F(\mathbf{x})\} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{\mathbf{k}\in\mathbb{Z}^q} f_n[\mathbf{k}]\, \psi(\mathbf{x} - \mathbf{k}) = \sum_{\mathbf{k}\in\mathbb{Z}^q} \left(\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_n[\mathbf{k}]\right) \psi(\mathbf{x} - \mathbf{k}) = \bar{f} \sum_{\mathbf{k}\in\mathbb{Z}^q} \psi(\mathbf{x} - \mathbf{k}) = \bar{f}$. Then, we can expand J_0 into $J_0 = 2\, \left(\bar{f}\right)^2 - 2 \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(\sum_{\mathbf{k}_1\in\mathbb{Z}^q} f_n[\mathbf{k}_1]\, \psi(\mathbf{k}_0 - \mathbf{k}_1)\right) \left(\sum_{\mathbf{k}_2\in\mathbb{Z}^q} f_n[\mathbf{k}_2]\, \psi(\mathbf{k}_0 + \Delta\mathbf{x} - \mathbf{k}_2)\right) = 2\, \left(\bar{f}\right)^2 - 2 \sum_{\mathbf{k}_1\in\mathbb{Z}^q} \sum_{\mathbf{k}_2\in\mathbb{Z}^q} \left(\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_n[\mathbf{k}_1]\, f_n[\mathbf{k}_2]\right) \psi(\mathbf{k}_0 - \mathbf{k}_1)\, \psi(\mathbf{k}_0 + \Delta\mathbf{x} - \mathbf{k}_2) = 2\, \left(\bar{f}\right)^2 - 2\, C \sum_{\mathbf{k}\in\mathbb{Z}^q} \psi(\mathbf{k}_0 - \mathbf{k})\, \psi(\mathbf{k}_0 + \Delta\mathbf{x} - \mathbf{k})$, where we have used the assumption of independence between samples. As ψ is interpolant, it finally stands out that $J_0(\Delta\mathbf{x}) = 2\, \left(\bar{f}\right)^2 - 2\, C\, \psi(\Delta\mathbf{x})$ is location-independent.

Please note that we did not have to use (7) in this derivation; meanwhile, the conclusion is that registering uncorrelated data to a translated version of themselves results in a SSD criterion that mirrors the shape of the basis function—literally so, since the plot of J_0 *versus* $\Delta\mathbf{x}$ is nothing but an upside-down version of ψ *versus* $\Delta\mathbf{x}$. As the optimizer is minimizing J in terms of $\Delta\mathbf{x}$, it is important that $(-\psi)$ exhibits as few local minima as possible to avoid confusing it; in addition, the funnel-like appearance of $(-\psi)$ should have an opening as widened as possible to further promote robustness. Among the bases that satisfy (7) and that were mentioned in this paper, $(-\mathrm{rect})$ is least favorable because of its deficiencies regarding its order of approximation, because it lacks a well-defined optimum, and because it has the narrowest capture range. The $(-\mathrm{sinc})$ case, in addition to its slow decay, has infinitely many nonglobal minima; moreover, its capture range is suboptimal, with a support smaller than $[-1.431, 1.431]$. So is $(-\phi_S)$, with a capture range of support smaller than $[-1.436, 1.436]$ and a less-than-optimal order of approximation. In conclusion, the best basis is that which we proposed in Figure 2, since it offers robustness thanks to both the absence of spurious optima and the largest capture range of support $[-\frac{3}{2}, \frac{3}{2}]$; this support is also the shortest one for a given order of approximation, resulting in computational efficiency and fidelity to data.

5. CONCLUSIONS

We have built a new family of interpolating bases characterized by a tunable order of approximation L. In general, these bases are not made of polynomial pieces; yet, the linear combination of their integer shifts allows the exact representation of all polynomials of degree lesser than L. This ensures that the outcome of the interpolation process is a function that precisely matches the L first terms of the Taylor expansion of any data, thus ensuring their high fidelity.

Our new family of bases satisfies a property known as constant variance, which emerges as an important requirement for image registration but which has been mainly overlooked so far. We propose an explicit construction where several degrees of freedom are singled out. These degrees of freedom can be exploited to further constrain the design, for example to minimize the support of the basis (which promotes computational efficiency), or to increase its regularity (which facilitates the task of gradient-based optimizers).

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