CMSC5743 2021F Homework 2

Due: Nov. 11, 2021

All solutions should be submitted to the blackboard in the format of **PDF/MS Word**.

Q1 (13%)

(a) (4%) Consider a formulation as follows.

$$\min_{\beta_1,\beta_2,\beta_3} \quad \frac{1}{2}\left\| \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} \right\|_2^2 + 3(|\beta_1| + |\beta_2| + |\beta_3|).$$

Please transfer above formulation as ADMM formulation form, that is

$$\begin{aligned} \min_{\boldsymbol{x},\boldsymbol{z}} \quad & f(\boldsymbol{x}) + g(\boldsymbol{z}) \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{z} = \boldsymbol{c}. \end{aligned}$$

Show that in your ADMM formulation form, $f(\cdot)$ and $g(\cdot)$ are convex.

(b) (7%) Please use ADMM to handle the formulation in (a). The stopping criterion is set to 2 iterations. All variables are initialized to be 0.

(c) (2%) Compare the coordinate descent method and ADMM to handle the formulation in (a) by discussing the advantages and disadvantages.

Q2 (12%)

(a) (4%) Considering the matrix $\boldsymbol{A}$:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 5 & 8 \\ -4 & 3 & -1 & 2 \end{bmatrix}.$$

Write down the singular values, and corresponding left and right singular vectors for $\boldsymbol{A}$.

(b) (4%) Show $\boldsymbol{A}$ in orthogonal rank 1 form, that is, show $\boldsymbol{A}$ as a sum of outer products that are mutually orthogonal.

(c) (4%) Show the 2-norm and the Frobenius norm of the error in replacing A by the rank 1 approximation in (b).

Q3 (13%)

(a) (4%) Assume $\boldsymbol{A} = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$ and rank is 2. Computing Singular Value Decomposition for $\boldsymbol{A}$, that is $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\top$.

(b) (4%) Given a matrix $\boldsymbol{A}$, prove that $\boldsymbol{A}\boldsymbol{A}^\top$ and $\boldsymbol{A}^\top\boldsymbol{A}$ have the same singular values.

(c) (5%) Given a matrix $\boldsymbol{A}$, prove that $\sigma_1 \geq |\lambda|_{\max}$, namely, its largest singular value dominates all eigenvalues.

Q4 (12%)

(a) (4%) Given $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}$. Calculate $\boldsymbol{A} \otimes \boldsymbol{B}$ and $\boldsymbol{B} \otimes \boldsymbol{A}$ (Kronecker product).

(b) (4%) Let us have a rank-1 tensor $\boldsymbol{\mathcal{X}} = \left[\begin{array}{cc|cc} 1 & \sqrt{2} & \sqrt{2} & 2 \\ \sqrt{2} & 2 & 2 & 2\sqrt{2} \end{array} \right]$. Calculating $\|\boldsymbol{\mathcal{X}}\|_F$ (Frobenius norm)

(c) (4%) Write down the 1-flattening of $\boldsymbol{\mathcal{X}}$ (1-flattening means only the first dim to flatten).

Q5 (12%) This exercise provide an example to show the benefit of SVD over Eigen-decomposition.

Suppose $\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Assume in real-world scenarios, due to some trouble, the $A[4, 1]$ entry changes from zero to $\frac{1}{60000}$ and mark the new matrix as $\boldsymbol{A}'$. Now $\boldsymbol{A}'$ is a full-rank matrix.

(a) (4%) Calculate the eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{A}'$.

(b) (4%) Calculate the singular value of $\boldsymbol{A}$ and $\boldsymbol{A}'$.

(c) (4%) What do you observe from the calculation results.

Q6 (13%) Construct a rank-1 matrix $\boldsymbol{A}$ satisfying all the following conditions.

- $\boldsymbol{A}\boldsymbol{v} = 12\boldsymbol{u}$;
- $\boldsymbol{v} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$;
- $\boldsymbol{u} = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

Q7 (12%) Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ be a matrix, and let $\boldsymbol{A}^\top$ be the transposed matrix of $\boldsymbol{A}$.

(a) (4%) Show that both $\boldsymbol{A}^\top\boldsymbol{A}$ and $\boldsymbol{A}\boldsymbol{A}^\top$ are positive semidefinite.

(b) (4%) Show that $\boldsymbol{A}^\top\boldsymbol{A}$ and $\boldsymbol{A}\boldsymbol{A}^\top$ have exactly the same nonzero eigenvalues.

(c) (4%) If we know that $m = n$ and $\boldsymbol{A}$ is positive semidefinite, show that the eigenvalues and singular values of $\boldsymbol{A}$ are exactly the same.

Q8 (13%) Assume that $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^n$ is a matrix, and $\mathrm{tr}(\cdot)$ is the trace function, *i.e.* $\mathrm{tr}(\boldsymbol{A}) = \sum_{i=1}^n a_{ii}$ is the sum of diagonal entries.

(a) (3%) Show that $\|\boldsymbol{A}\|_F^2 = \mathrm{tr}(\boldsymbol{A}^\top\boldsymbol{A})$.

(b) (3%) Assume that $\boldsymbol{M} \in \mathbb{R}^n$ is positive semidefinite, show that $\text{tr}(\boldsymbol{M}) = \sum_{i=1}^n \lambda_i$, where $\lambda_1, \cdots, \lambda_n$ are eigenvalues of $\boldsymbol{M}$.

(c) (7%) (**Hard**) You are given the following inequality,

$$\text{tr}(\boldsymbol{AB}) \leq \sum_{i=1}^{n} \sigma_i(\boldsymbol{A})\sigma_i(\boldsymbol{B}),$$

where $\sigma_1(\boldsymbol{A}) \geq \cdots \geq \sigma_n(\boldsymbol{A})$ and $\sigma_1(\boldsymbol{B}) \geq \cdots \geq \sigma_n(\boldsymbol{B})$ are singular values of $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively. Find an optimal solution to the following low rank approximation,

$$\min_{\substack{\boldsymbol{X} \in \mathbb{R}^{m \times n} \\ \text{rank}(\boldsymbol{X}) \leq k}} \|\boldsymbol{X} - \boldsymbol{Y}\|_F^2,$$

where matrix $\boldsymbol{Y} \in \mathbb{R}^{m \times n}$ is fixed and has a full rank. The rank upper bound $k \leq \min\{m, n\}$. (Hint: use the results in the above questions, and consider singular values of the two matrices.)