CENG4480 2021F Homework 3

Solutions

Q1 (10%)

Assume two normal random variables X_1, X_2, and $X_1 \sim N(\mu_1, \sigma_1)$, $X_2 \sim N(\mu_2, \sigma_2)$. A new random variable X_3 is the weighted sum of these two normal random variables (i.e. $aX_1 + bX_2$). Please answer the following two questions:

(a) If X_1 andX_2 are independent, try to deduce the expectation and variance of X_3.

(b) If X_1 andX_2 are not independent and the covariance $Cov(X_1, X_2) = \sigma_{12}$, try to deduce the expectation and variance of X_3.

A1 (1)

$$\mu_3 = a\mu_1 + b\mu_2 \tag{1}$$

$$\sigma_3 = a^2\sigma_1 + b^2\sigma_2 \tag{2}$$

(2)

$$\mu_3 = a\mu_1 + b\mu_2 \tag{3}$$

$$\sigma_3 = a^2\sigma_1 + b^2\sigma_2 + 2 * ab\sigma_{12} \tag{4}$$

Q2 (10%)

Given a linear system

$$\begin{cases} \boldsymbol{x}_t = \boldsymbol{A}_{t-1}\boldsymbol{x}_{t-1} + \boldsymbol{\omega}_{t-1}, \\ \boldsymbol{z}_t = \boldsymbol{B}_t\boldsymbol{x}_t + \boldsymbol{v}_t, \\ \boldsymbol{v}_t = \boldsymbol{C}_{t-1}\boldsymbol{v}_{t-1} + \boldsymbol{n}_{t-1}, \end{cases} \tag{5}$$

where $\boldsymbol{\omega}_t$ and $\boldsymbol{n}_t$ are independent and obey Gaussian distribution zero-mean and covariance $\boldsymbol{Q}_t$ and $\boldsymbol{R}_t$, respectively. Please give the estimate equation and measurement equation of the system.

A2

$$\begin{pmatrix} \boldsymbol{x}_t \\ \boldsymbol{v}_t \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_{t-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{C}_{t-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_{t-1} \\ \boldsymbol{v}_{t-1} \end{pmatrix} + \begin{pmatrix} \boldsymbol{\omega}_{t-1} \\ \boldsymbol{n}_{t-1} \end{pmatrix} \tag{6}$$

$$\boldsymbol{z}_t = \begin{pmatrix} \boldsymbol{B}_t & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_t \\ \boldsymbol{v}_t \end{pmatrix} \tag{7}$$

Q3 (20%)

(a) Give the five equation of Kalman Filter. Explain the meaning of P, Q and R in the Kalman Filter.

(b) Here is an example to help you understand Kalman Filter application. At time $t-1$, the distance between a robot and a wall is 10 meters and its velocity is $4m/s$. What's the estimate distance at time t? If you use the radar to measure the distance and the result is $7m$, which one would you believe? If I told you that the accuracy of radar is 80% and the accuracy of your estimate distance is 90%, what's your final answer based on Kalman Filter?

A3 **(a)** Refer to Page 8 and Page 9, Lecture 08

$$\begin{aligned}
\boldsymbol{x}_t^- &= \boldsymbol{A}_t\boldsymbol{x}_{t-1} + \boldsymbol{B}_t\boldsymbol{u}_t \\
\boldsymbol{P}_t^- &= \boldsymbol{A}_t\boldsymbol{P}_{t-1}\boldsymbol{A}_t^\top + \boldsymbol{Q}_t \\
\boldsymbol{x}_t &= \boldsymbol{x}_t^- + \boldsymbol{K}_t\left(\boldsymbol{z}_t - \boldsymbol{C}\boldsymbol{x}_t^-\right) \\
\boldsymbol{P}_t &= \boldsymbol{P}_t^- - \boldsymbol{K}_t\boldsymbol{C}\boldsymbol{P}_t^- \\
\boldsymbol{K}_t &= \boldsymbol{P}_t^-\boldsymbol{C}^\top\left(\boldsymbol{C}\boldsymbol{P}_t^-\boldsymbol{C}^\top + \boldsymbol{R}_t\right)^{-1}
\end{aligned} \tag{8}$$

(b)

$$\text{result} = (1 - \frac{0.9}{0.9+0.8}) * 6 + \frac{0.9}{0.9+0.8} * 7 = 6.52m \tag{9}$$

$$\text{Kalman Gain} = \frac{0.9}{0.9+0.8} \tag{10}$$

Q4 (20%)

The general equation of a liner estimate system is like $\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{w}_{t+1}$. Consider the following three different systems:

(a) $x(t) = 1.52x(t-1) + \omega_t$

(b) $x(t) = 1.52x(t-1) + 0.4x(t-2) + \omega_t$

Kalman Filter is used to estimate $x(t)$ (here $x(t)$ is a scalar). Please give the formulations of state transition matrix $\mathbf{A}$ and noise vector $\mathbf{w}_t$.

A4 For (a) system: $\mathbf{x}_{t+1} = x(t)$, $\mathbf{x}_t = x(t-1)$, $\mathbf{A} = 1.52$ and $\mathbf{w}_{t+1} = \omega_t$ (for this system, $\mathbf{x}_{t+1}$, $\mathbf{x}_t$, $\mathbf{A}$ and $\mathbf{w}$ are a scalar, respectively).

For (b) system:

$$\begin{cases} x(t-1) = 0 \cdot x(t-2) + 1 \cdot x(x-1) + 0 \\ x(t) = 0.4 \cdot x(t-2) + 1.52 \cdot x(t-1) + \omega_t. \end{cases}$$

Its matrix form is

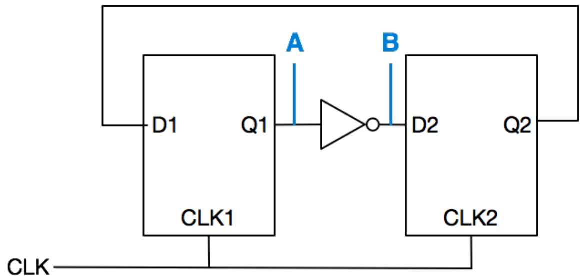
$$\begin{bmatrix} x(t-1) \\ x(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.4 & 1.52 \end{bmatrix} \cdot \begin{bmatrix} x(t-2) \\ x(t-1) \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_t \end{bmatrix}.$$

Therefore, $\mathbf{x}_t = \begin{bmatrix} x(t-2) \\ x(t-1) \end{bmatrix}$, $\mathbf{x}_{t+1} = \begin{bmatrix} x(t-1) \\ x(t) \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0.4 & 1.52 \end{bmatrix}$, and $\mathbf{w}_{t+1} = \begin{bmatrix} 0 \\ \omega_t \end{bmatrix}$.

Q5 (20%)

A digital clock is important in circuit design. Please answer the following **three** questions.

(a) Given the following circuit, CLK1 = CLK2 = 20MHz; Tff = 5ns; Tsetup = 5ns. The gate delay TG = 10ns. Please calculate the time margin. Note: Tff= delay of a flip flop, Tsetup=setup time of a flip flop, and TG is delay of a gate.



(b) In the above circuit, currently there is already one delay gate with delay TG. How many more similar delay gates can you insert between A and B without creating error?

(c) Sometimes we can take advantage of clock skew. For the above circuit, if the delay from CLK to CLK2 is 2ns, calculate the minimal clock period of the clock CLK.

A5

1. Period = $\frac{1}{20\text{M}} = 50\text{ns}$. Margin = $50 - 5 - 5 - 10 = 30\text{ns}$
2. Max #gate = $\frac{30}{10} = 3$. So 3 more gates can be inserted.
3. Since we are searching for minimal period, no slack is desired.

$$\text{Tc1} + \text{Tff} + \text{TG} + \text{Tsetup} = \text{TCLK} + \text{Tc2} \tag{11}$$

From Equation (11) we have $\text{TCLK} = 20 + \text{Tc1} - \text{Tc2}$. We set Tc1 = 0, then we can obtain TCLK = 18ns

Q6 (20%)

(a) Write down the differences between SRAM and DRAM .

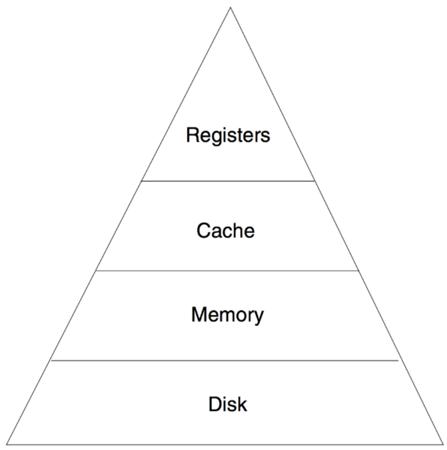
(b) Show the memory hierarchy and point out where are DRAM and SRAM employed respectively.

(a)

SRAM	DRAM
SRAM has lower access time, which is faster compared to DRAM.	DRAM has a higher access time. It is slower than SRAM.
SRAM is costlier than DRAM.	DRAM cost is lesser compared to SRAM.
SRAM needs a constant power supply, but it consumes less power.	DRAM requires more power consumption as the information is stored in the capacitor.
SRAM is of a smaller size.	DRAM is available in larger storage capacity.

A6 **(b)**

DRAM is usually employed in Main Memory while SRAM is employed in Cache.


Registers
Cache
Memory
Disk