NEAR-OPTIMAL MAP INFERENCE FOR DETERMINANTAL POINT PROCESSES

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IMAGE SEARCH: "JAGUAR"

Relevance only:

...

Relevance + diversity:

...

TASK: SUBSET SELECTION

two goals

quality

diversity

MATCHED SUMMARIZATION

Task:

Given a set of documents, select a set of doc pairs such that:
1) the pairs are high-quality (docs within a pair are similar), and
2) the overall set of pairs is diverse.

MATCHED SUMMARIZATION

Ground set: All possible (old, new) pairs.

- **Old (topic = bailout): Let Detroit go bankrupt.**
- **New (topic = bailout) : I'm not willing to sit back and say 'Too bad for Michigan'.**

- **Old (topic = bailout): Let Detroit go bankrupt.**
- **New (topic = abortion): The right next step ... is to see Roe vs Wade overturned.**

Quality only:

- **Old (topic = bailout): Let Detroit go bankrupt.**
- **New (topic = bailout): I'm not willing to sit back and say 'Too bad for Michigan'.**

- **Old (topic = bailout): I think there is need for economic stimulus.**
- **Old (topic = bailout): I have never supported the President's recovery act.**

- **Old (topic = bailout): TARP ought to be ended.**
- **New (topic = bailout): TARP got paid back and it kept the financial system from collapsing.**

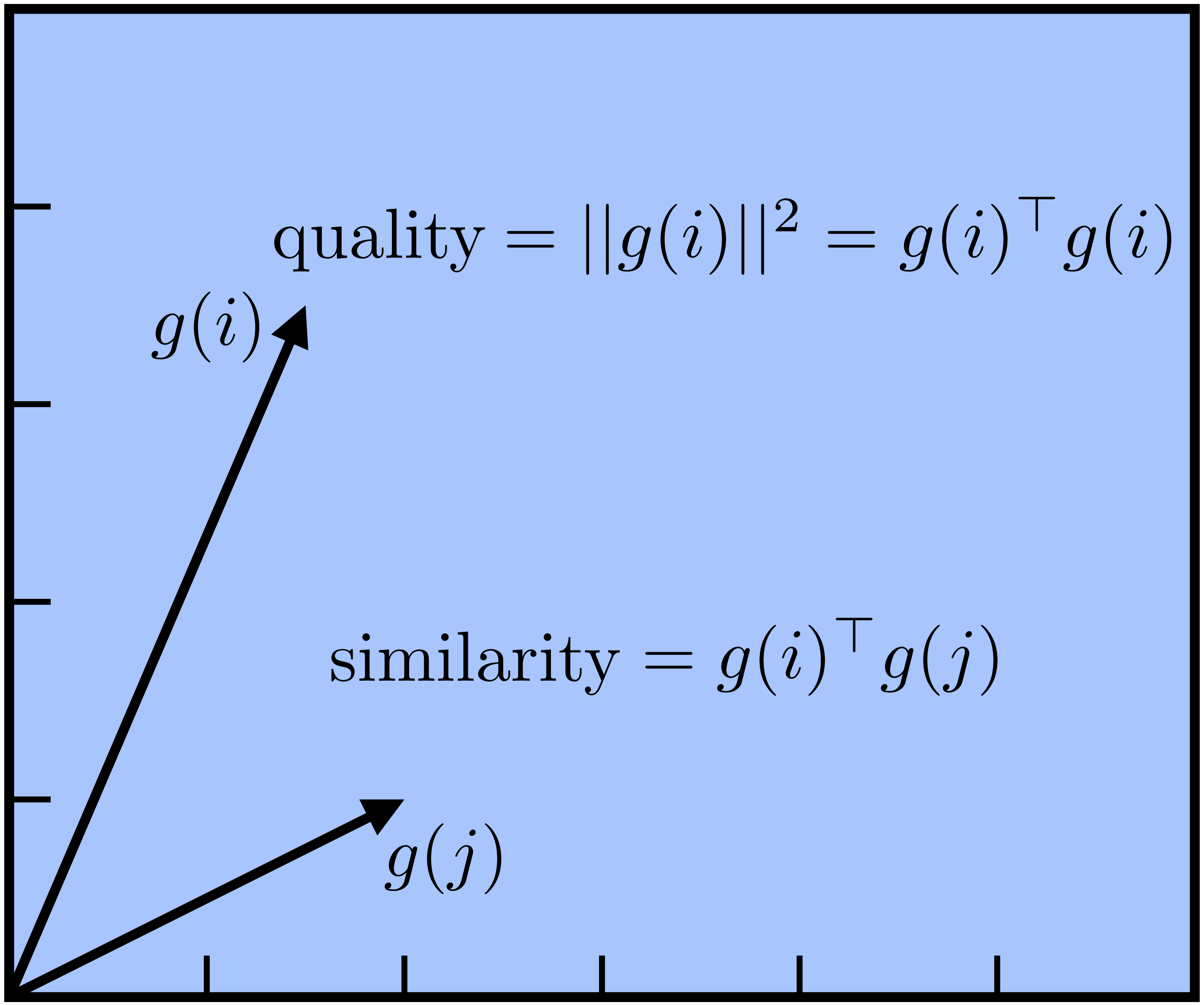
Quality + diversity:

- **Old (topic = bailout): Let Detroit go bankrupt.**
- **New (topic = bailout): I'm not willing to sit back and say 'Too bad for Michigan'.**

- **Old (topic = abortion): I will preserve and protect a woman's right to choose.**
- **New (topic = abortion): The right next step ... is to see Roe vs Wade overturned.**

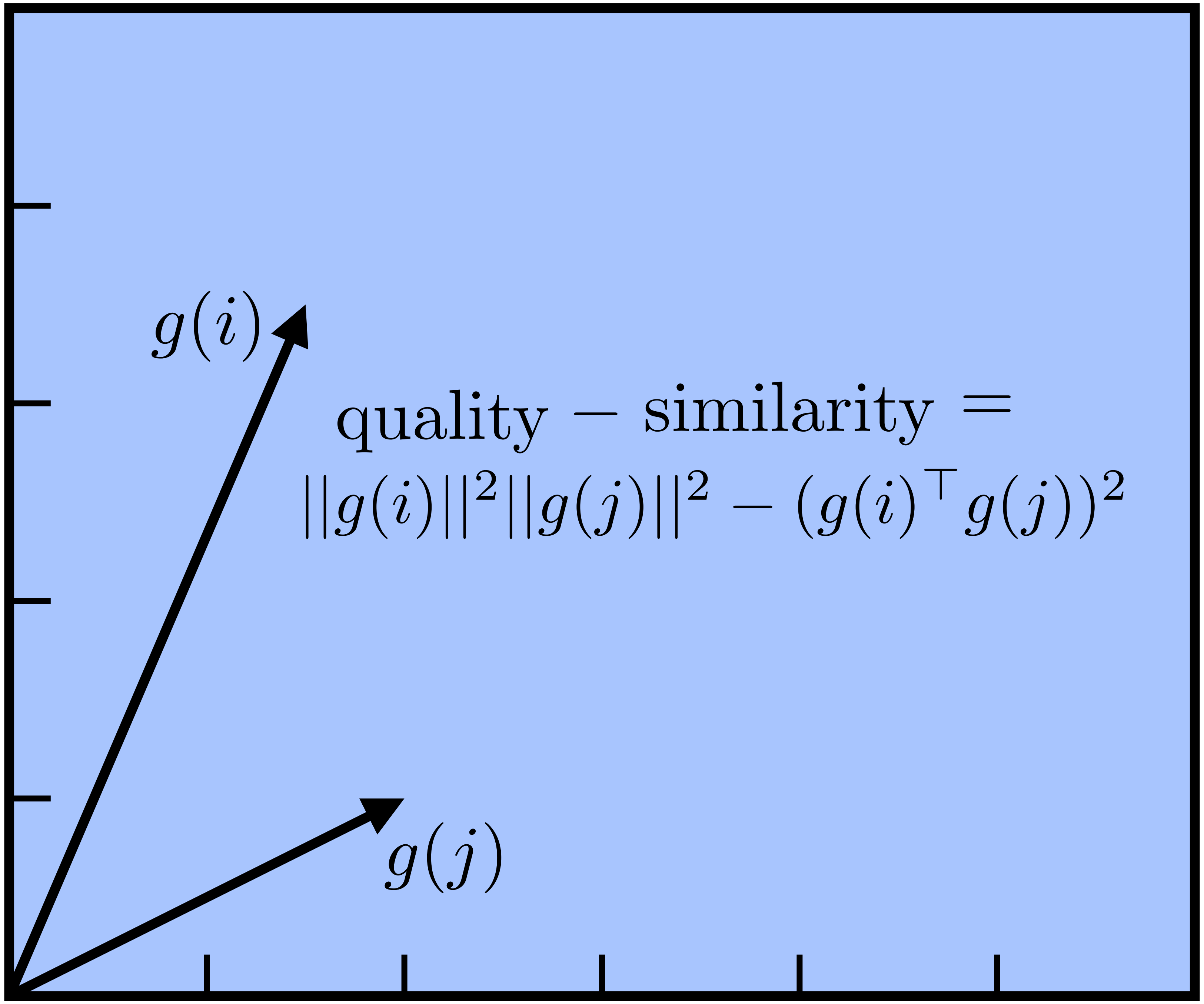
- **Old (topic = gun control): I just signed a major piece of legislation extending the ban on certain assault weapons.**
- **New (topic = gun control): I do not support any new legislation of an assault weapon ban nature.**

FORMALIZING



feature space

FORMALIZING



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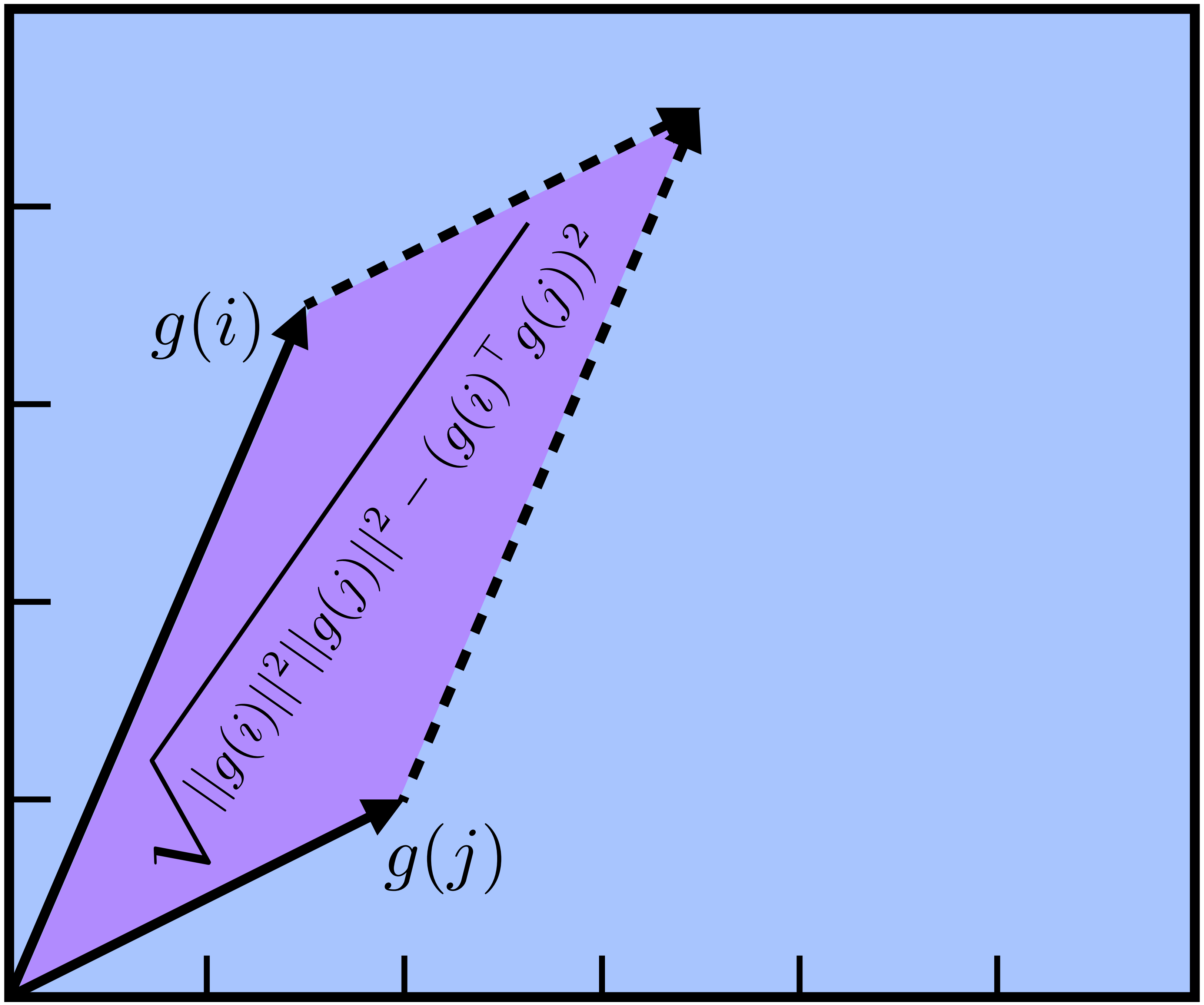

feature space

AREA AS A DET

$$||g(i)||^2 ||g(j)||^2 - (g(i)^\top g(j))^2$$

$$= \det \begin{pmatrix} ||g(i)||^2 & g(i)^\top g(j) \\ g(i)^\top g(j) & ||g(j)||^2 \end{pmatrix}$$

$$\begin{matrix} \text{——} g(1) \text{——} \\ \text{——} g(2) \text{——} \\ \vdots \\ \text{——} g(N) \text{——} \end{matrix}$$

AREA AS A DET

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$$= \det \begin{pmatrix} ||g(i)||^2 & g(i)^\top g(j) \\ g(i)^\top g(j) & ||g(j)||^2 \end{pmatrix}$$

$$= \det \left(\begin{bmatrix} \text{---} \; g(i) \; \text{---} \\ \text{---} \; g(j) \; \text{---} \end{bmatrix} \begin{bmatrix} | & | \\ g(i) & g(j) \\ | & | \end{bmatrix} \right)$$

VOLUME AS A DET

$$\text{“goodness” of } \{i,j\} = \text{quality \& diversity of } \{i,j\}$$

$$\propto \ \text{area}(\{i,j\})^2$$

$$\text{volume}_1 = \text{length}$$
$$\text{volume}_2 = \text{area}$$
$$\text{volume}_3 = \text{3D-volume}$$

$$|Y| = d \rightarrow \text{volume}_d(Y)^2 \ \propto \ \det((GG^\top)_Y)$$

$$\text{for positive semi-definite } L = GG^\top$$

$$\mathcal{P}(Y) \propto \det(L_Y)$$

VOLUME AS A DET

$$\text{“goodness” of } \{i,j\} = \text{quality \& diversity of } \{i,j\}$$
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Determinantal Point Process

$$|Y| = d \ \rightarrow \text{volume}_d(Y)^2 \ \propto \ \det((GG^\top)_Y)$$

$$\text{for positive semi-definite } L = GG^\top$$
$$\mathcal{P}(Y) \propto \det(L_Y)$$

VOLUME AS A DET

$$\text{“goodness” of } \{i,j\} = \text{quality \& diversity of } \{i,j\}$$
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DPP

$$|Y| = d \ \rightarrow \text{volume}_d(Y)^2 \ \propto \ \det((GG^\top)_Y)$$

$$\text{for positive semi-definite } L = GG^\top$$
$$\mathcal{P}(Y) \propto \det(L_Y)$$

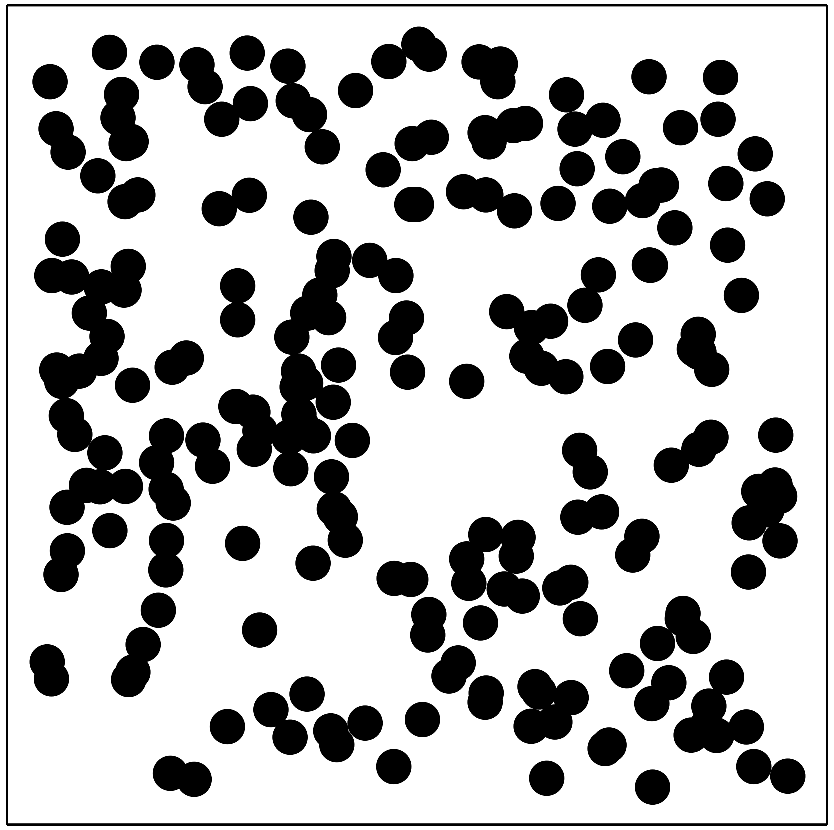
DPP INFERENCE

- Exact and efficient $O(N^3)$
  - normalization: $\sum_Y \det(L_Y) = \det(L + I)$
  - marginalization: $\mathcal{P}(A \subseteq Y)$
  - conditioning: $\mathcal{P}(A \mid B \subseteq Y)$
  - sampling: $Y \sim \mathcal{P}(Y) \propto \det(L_Y)$

DPP INFERENCE

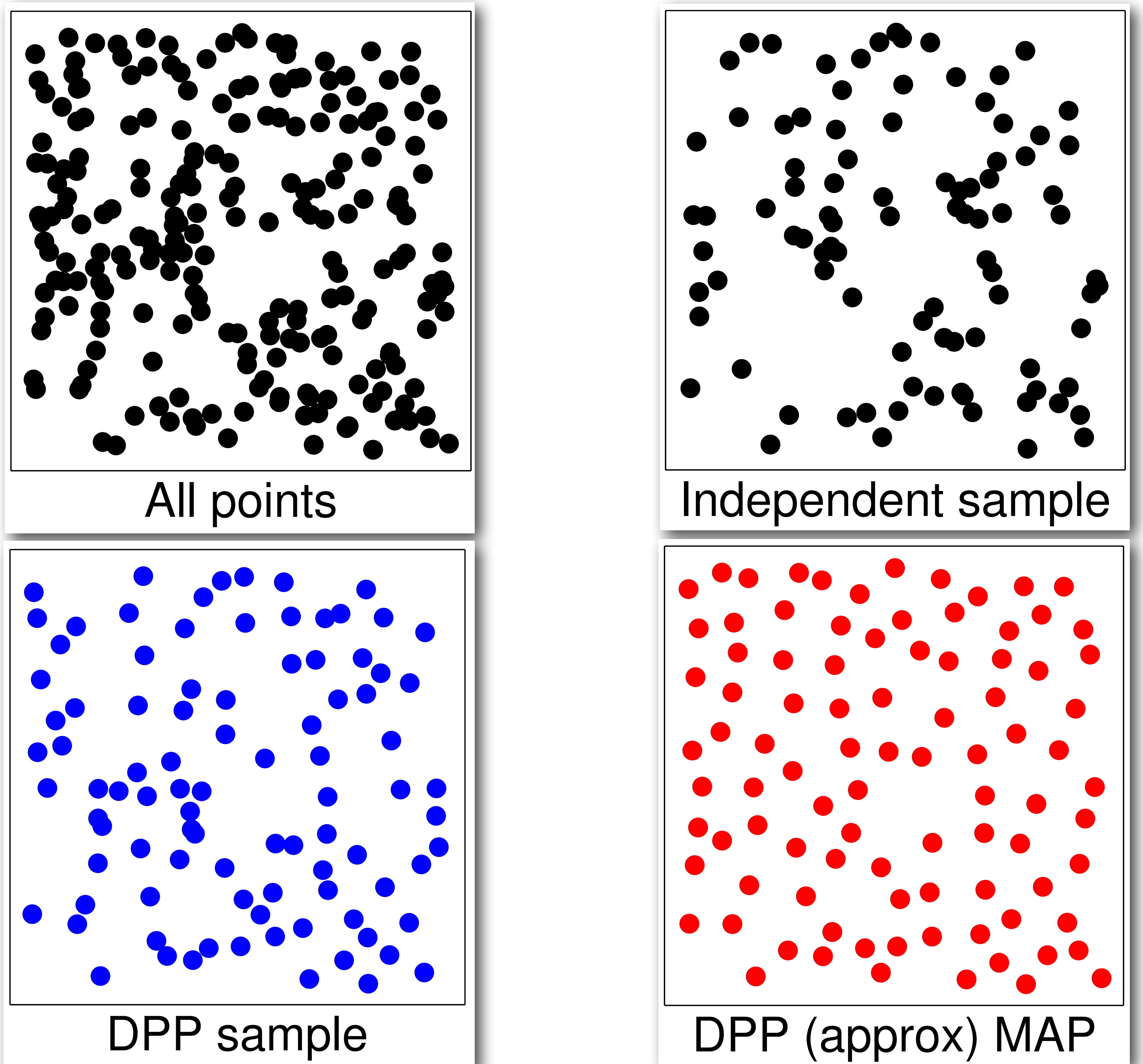
What about DPP MAP?

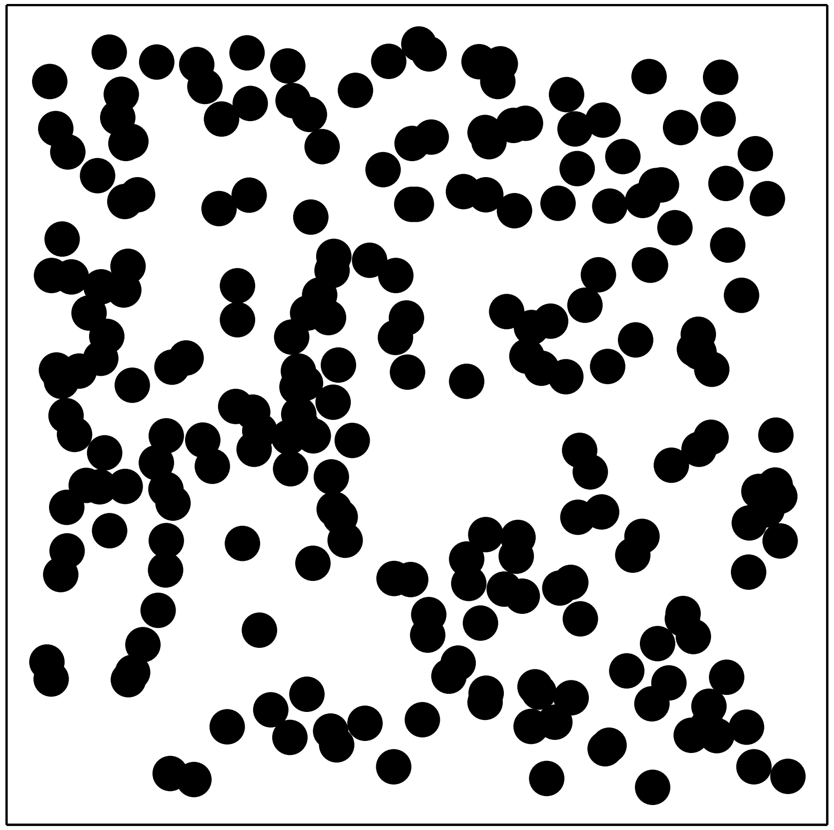
$$\arg\max_Y \det(L_Y)$$



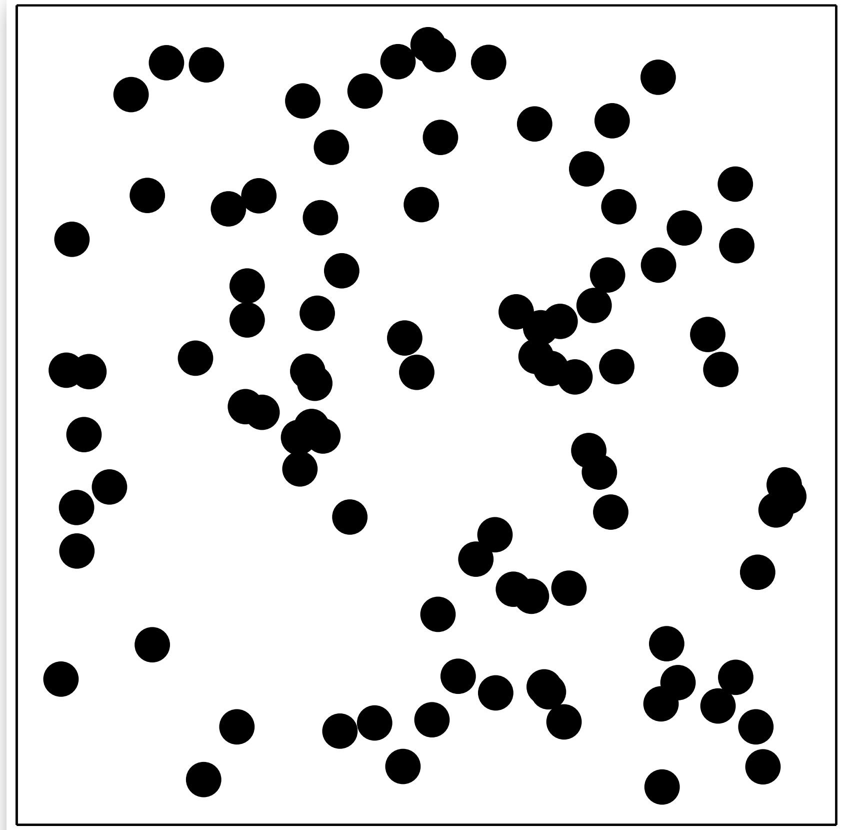
All points

$$g(i)^\top g(j) = L_{ij} = \exp(-||p_i - p_j||^2)$$

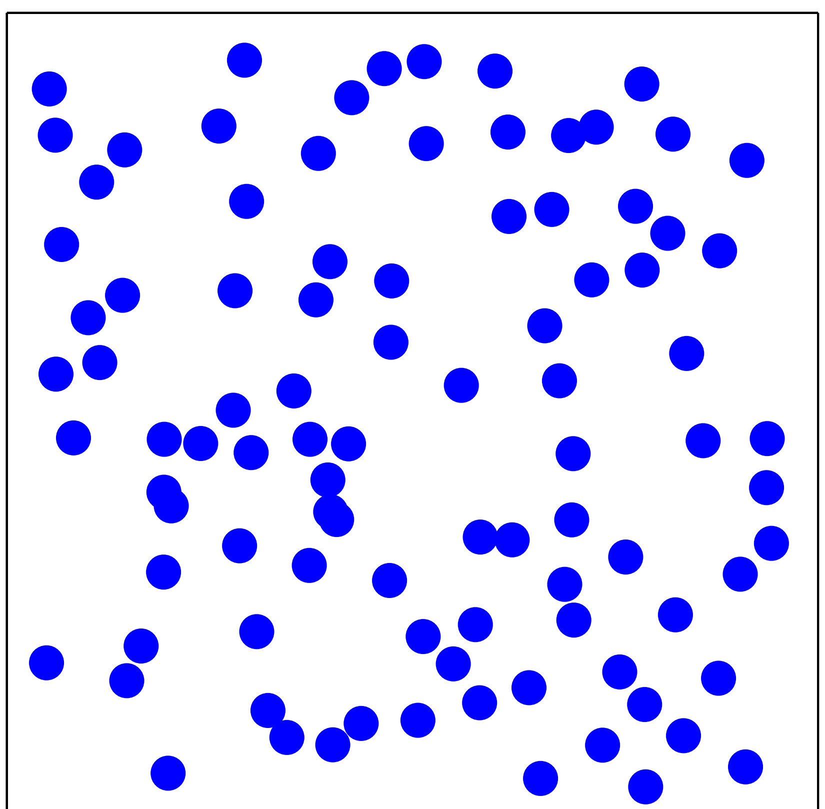

All points
Independent sample
DPP sample
DPP (approx) MAP



All points



Independent sample



DPP sample

DPP (approx) MAP

SUBMODULARITY TO THE RESCUE

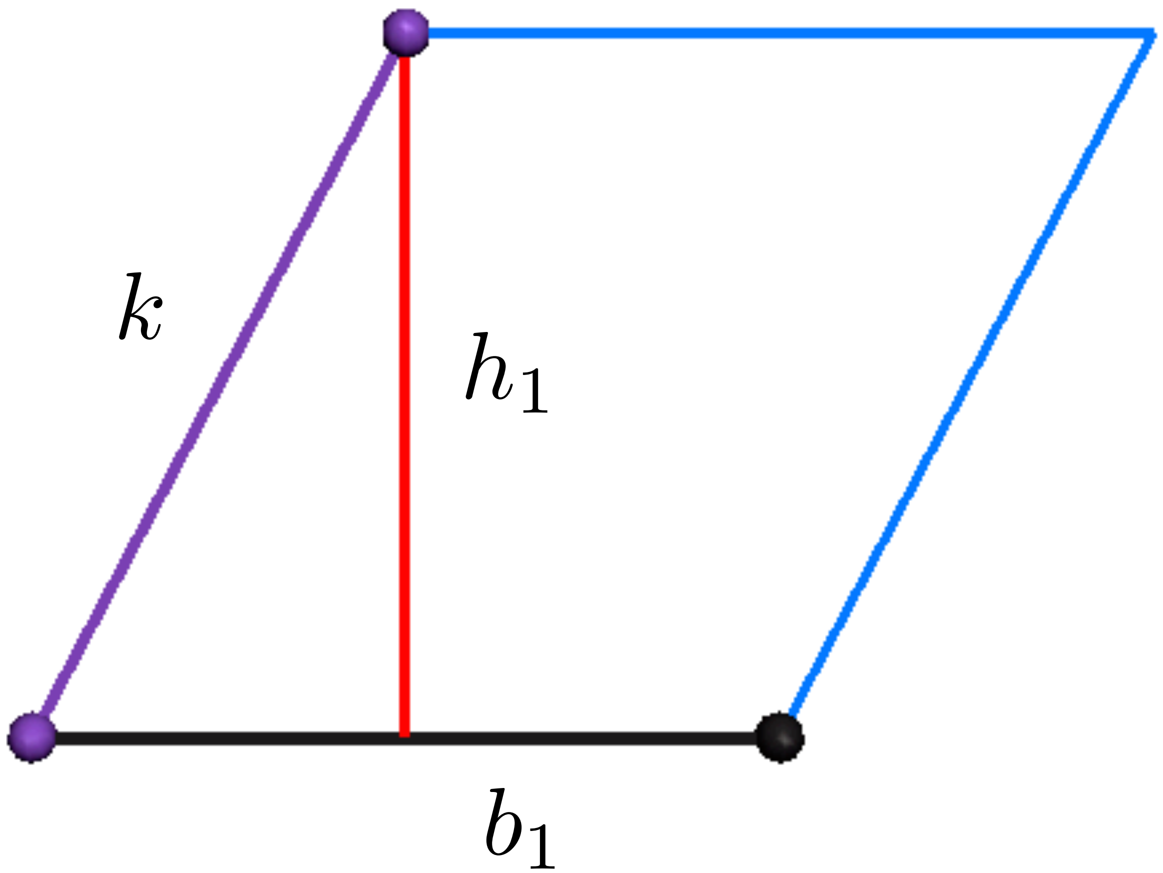
$$f(Y) = \det(L_Y) \text{ is log-submodular}$$

Diminishing returns:

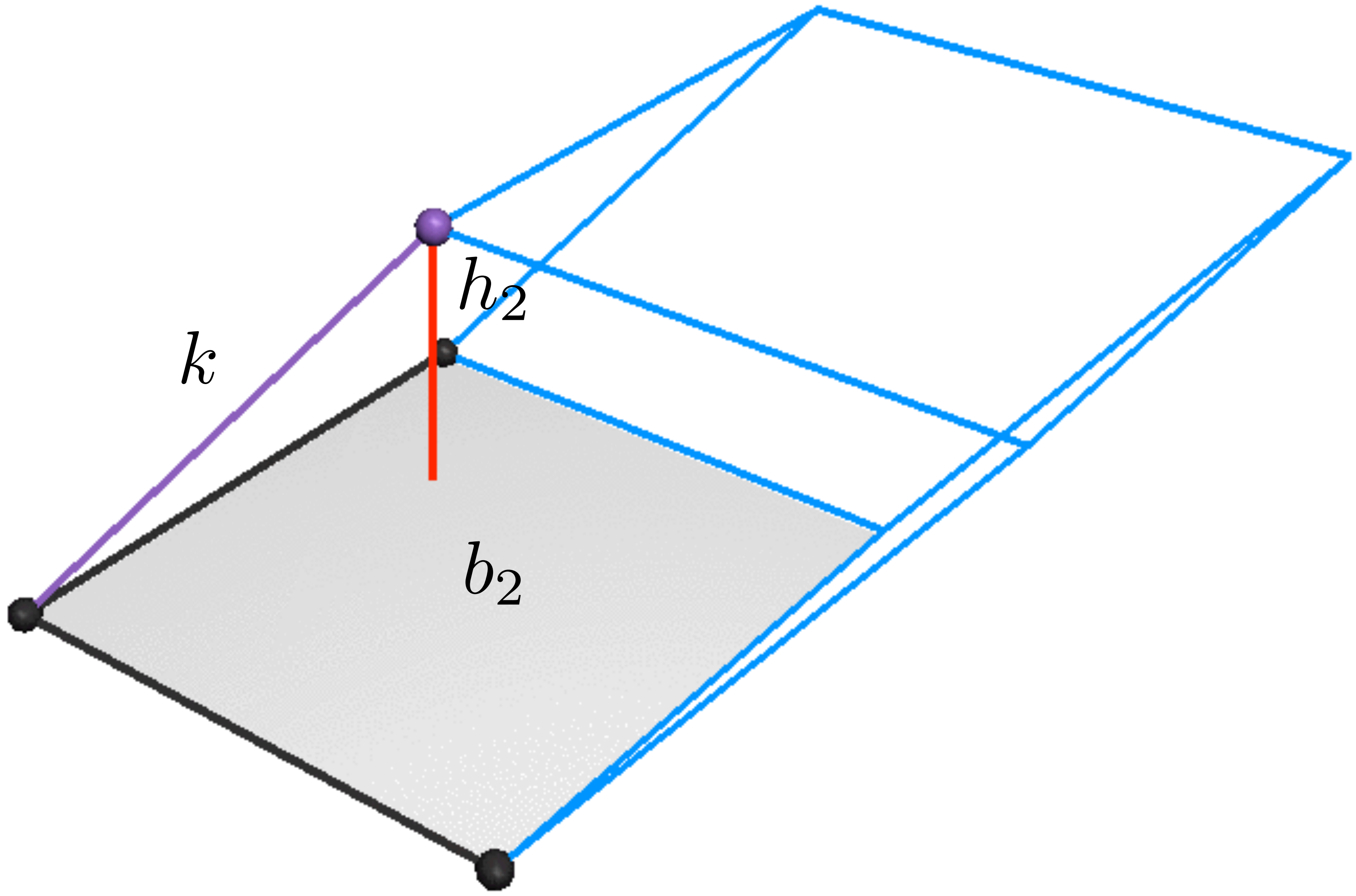
$$\frac{f(Y \cup \{k\})}{f(Y)} \leq \frac{f(X \cup \{k\})}{f(X)}$$

$$X \subseteq Y,\ k \notin Y$$

$$\frac{\mathrm{vol}(X \cup \{k\})}{\mathrm{vol}(X)} = \frac{b_1 h_1}{b_1} = h_1$$

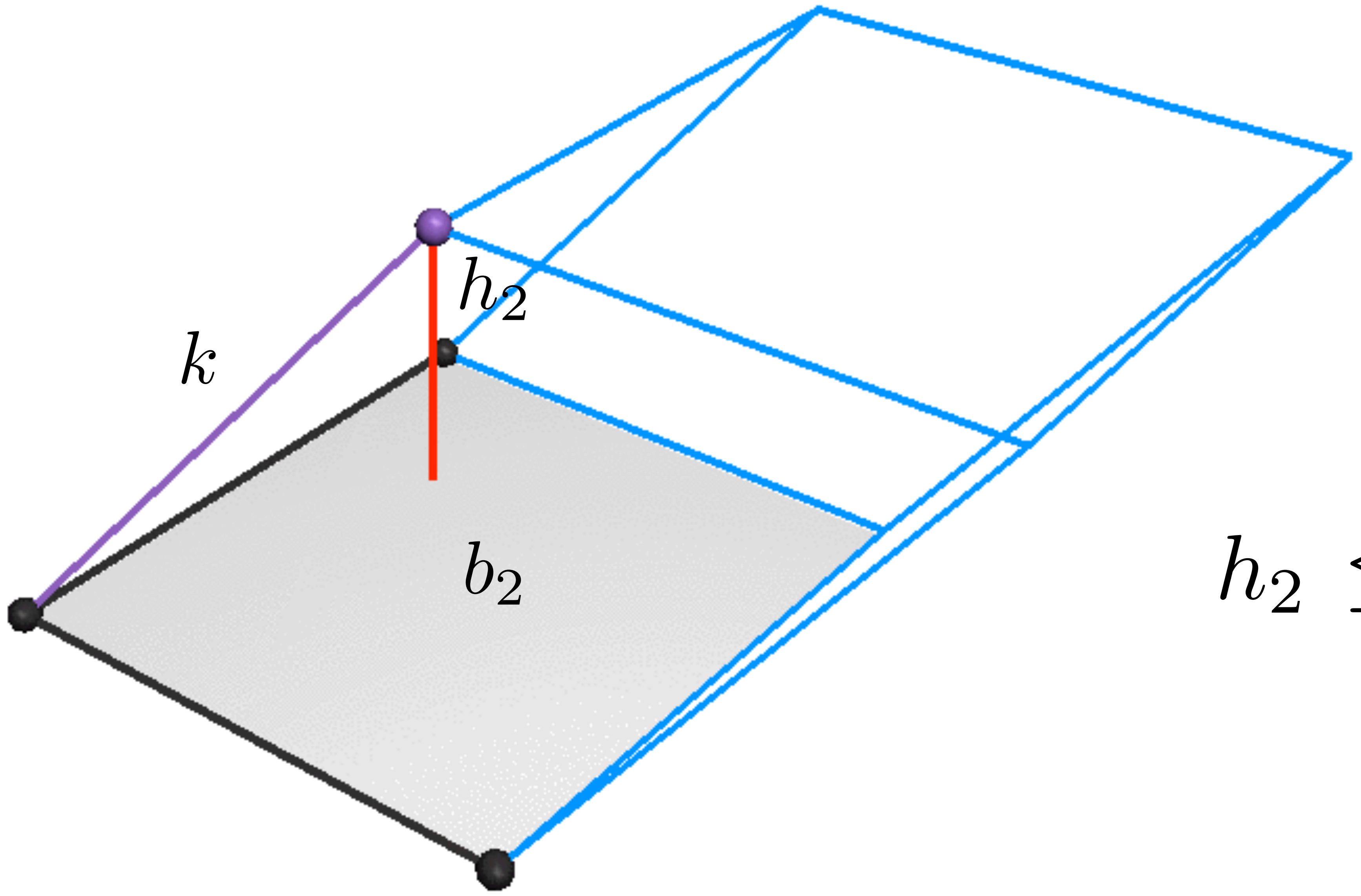


$$\frac{\mathrm{vol}(X \cup \{k\})}{\mathrm{vol}(X)} = \frac{b_1 h_1}{b_1} = h_1$$



$$\frac{\mathrm{vol}(Y \cup \{k\})}{\mathrm{vol}(Y)} = \frac{b_2 h_2}{b_2} = h_2$$

$$\frac{\mathrm{vol}(X \cup \{k\})}{\mathrm{vol}(X)}$$



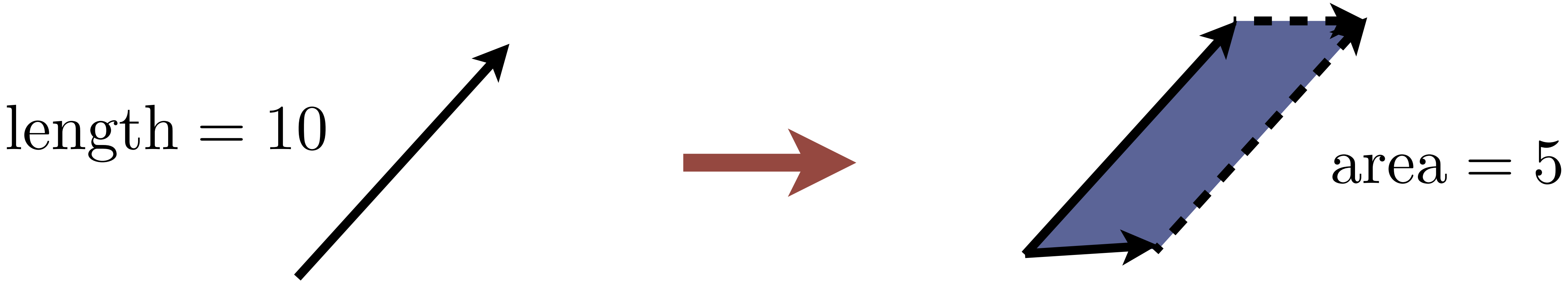
$$h_2 \leq h_1$$

$$\frac{\mathrm{vol}(Y \cup \{k\})}{\mathrm{vol}(Y)}$$

MONOTONICITY

$$X \subseteq Y \implies f(X) \leq f(Y)$$

Det is non-monotone: $\det(L_X) > \det(L_Y)$ for some X, Y



PRIOR WORK

Monotone:
"greedy" (1 - 1/e)-approx
Nemhauser and Wolsey (1978)

Non-monotone:
"symmetric greedy" 1/2-approx
Buchbinder et al. (2012)

Performs poorly in practice

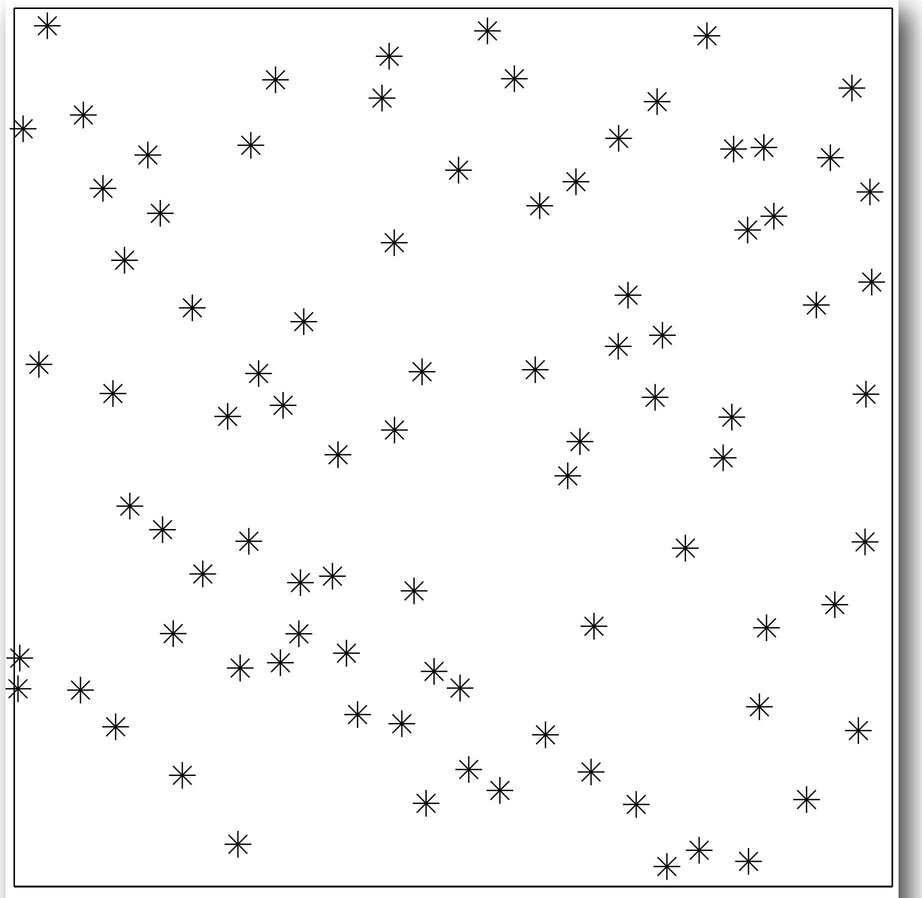
Non-monotone + constraints:
"multilinear" 1/4-approx sans constraints,
various (lesser) guarantees dependent on constraint type
Chekuri et al. (2011)

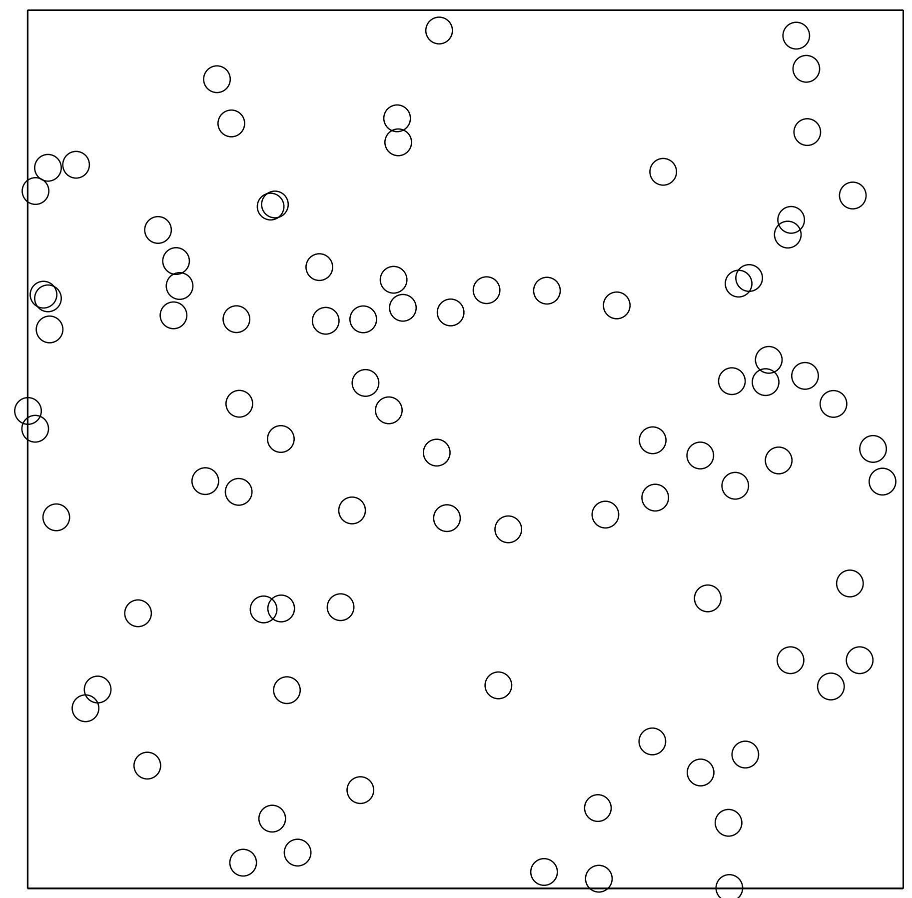
PRIOR WORK

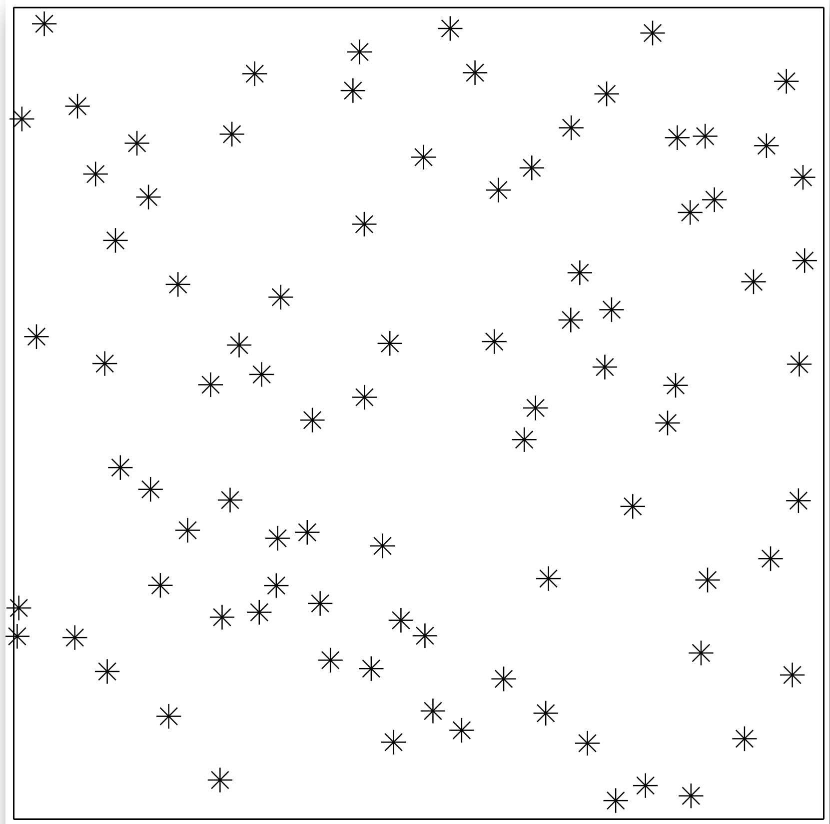
Non-monotone + constraints:
"multilinear" 1/4-approx sans constraints,
various (lesser) guarantees dependent on constraint type
Chekuri et al. (2011)

$\arg\max_Y \det(L_Y)$ → $\arg\max_{Y \in S} \det(L_Y)$
where S is a solvable polytope

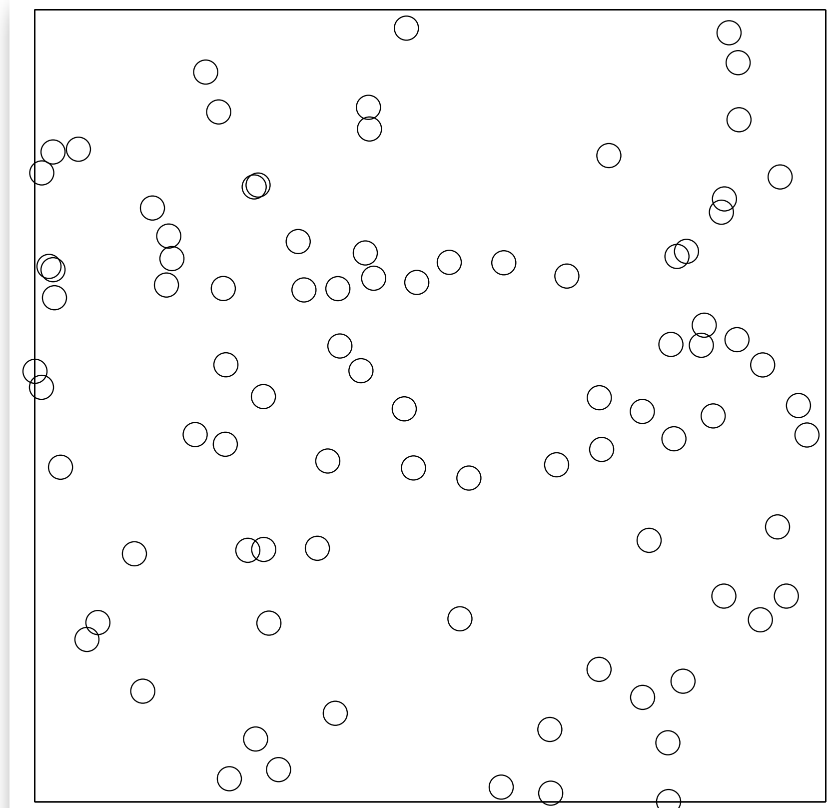
IMAGE COMPARISON WITH CONSTRAINTS


Point set 1

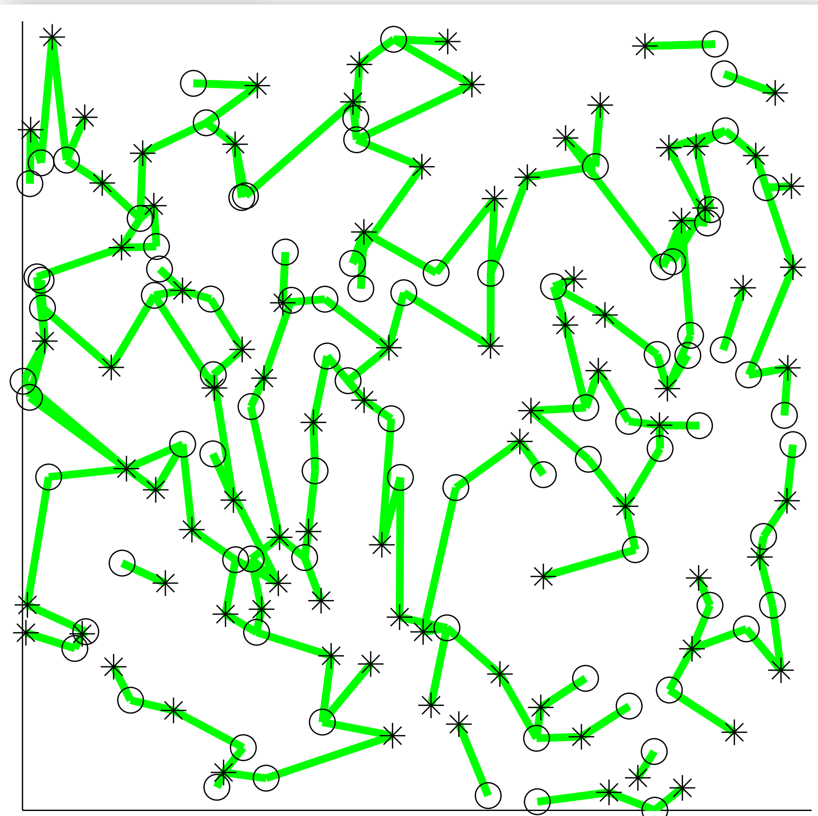

Point set 2



Point set 1

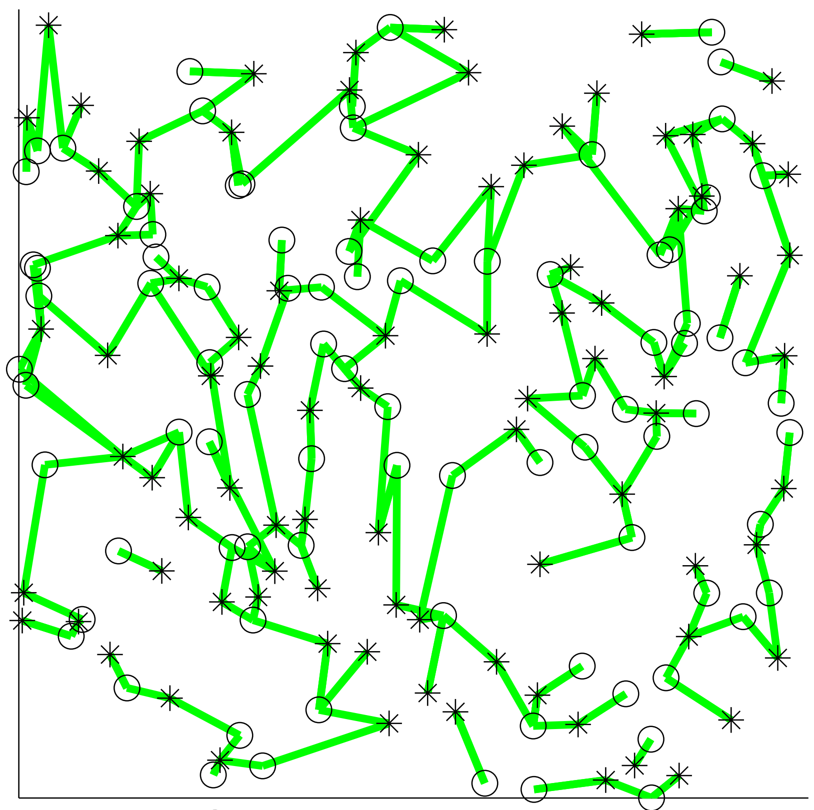


Point set 2

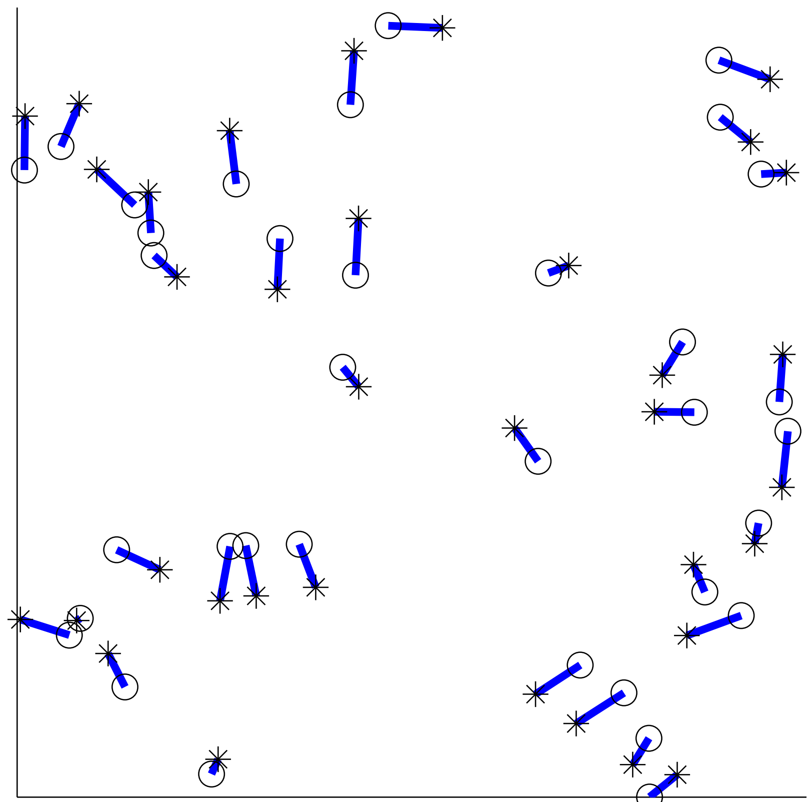


Ground set of pairs

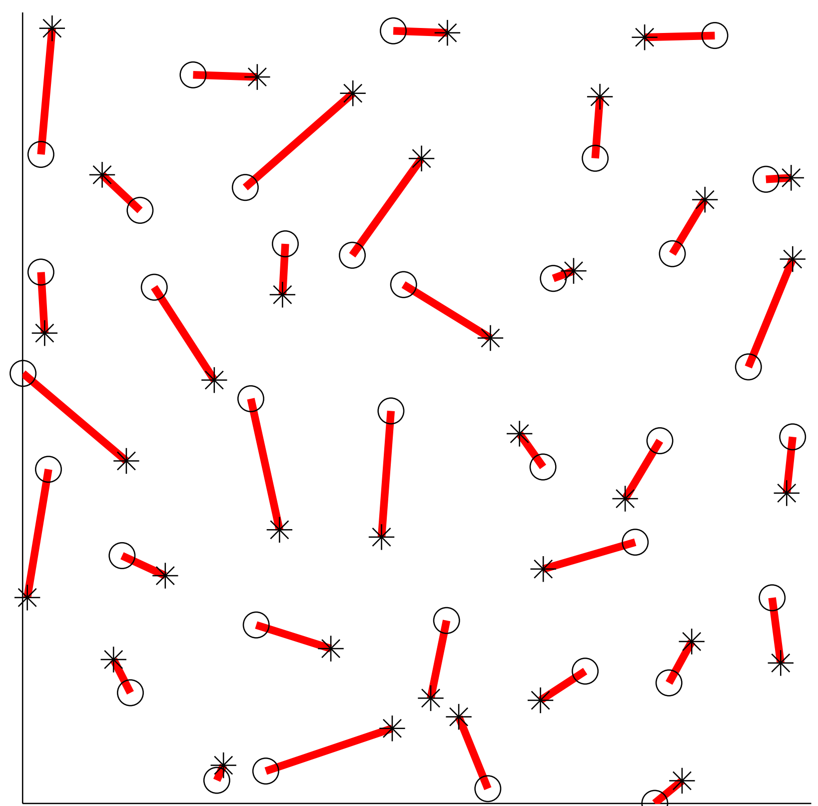
$Y \in$ matching polytope



Ground set of pairs



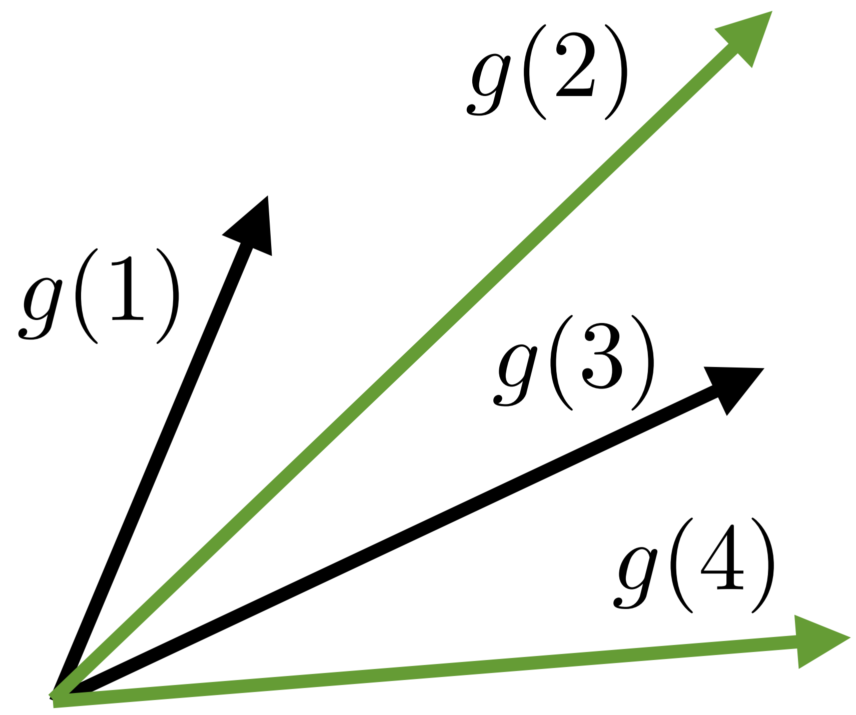
Highest quality 1:1 matching



DPP MAP 1:1 matching

CHEKURI ET AL. 2011

Step 1: Relax inclusion-exclusion



$$Y = \{2, 4\}$$

$$\mathbf{x} = [0, 1, 0, 1]$$

$$x_i \in [0, 1]$$

CHEKURI ET AL. 2011

Step 2: Extend objective

$$F(\mathbf{x}) = E_{\mathbf{x}}[\log f(Y)]$$ multilinear extension

log-submodular, like $\det(L_Y)$

CHEKURI ET AL. 2011

Step 2: Extend objective

$$F(\mathbf{x}) = E_{\mathbf{x}}[\log f(Y)]$$ multilinear extension

$$= \sum_Y \boxed{\prod_{i \in Y} x_i \prod_{i \notin Y} (1 - x_i)} \log f(Y)$$

$\downarrow$

$$p_Y$$

CHEKURI ET AL. 2011

Step 2: Extend objective

$$F(\mathbf{x}) = E_{\mathbf{x}}[\log f(Y)]$$

multilinear extension

$$= \sum_Y \prod_{i \in Y} x_i \prod_{i \notin Y} (1 - x_i) \log f(Y)$$

2^N subsets

CHEKURI ET AL. 2011

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2^N subsets

Step 3: Optimize using gradient-based methods $\frac{\partial F(\mathbf{x})}{\partial \mathbf{x}}$

Step 4: If unconstrained, solution will already be integer; else, round solution: $x_i \in [0,1] \to x_i \in \{0,1\}$

CHEKURI ET AL. 2011

Step 2: Extend objective

$$F(\mathbf{x}) = E_{\mathbf{x}}[\log f(Y)]$$ multilinear extension

$$= \sum_Y \prod_{i \in Y} x_i \prod_{i \notin Y} (1 - x_i) \log f(Y)$$

2^N subsets $\implies$ Monte Carlo required

Step 3: Optimize using gradient-based methods $\frac{\partial F(\mathbf{x})}{\partial \mathbf{x}}$

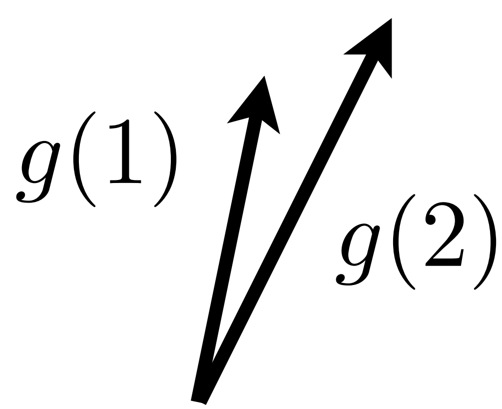
Step 4: If unconstrained, solution will already be integer; else, round solution: $x_i \in [0, 1] \to x_i \in \{0, 1\}$

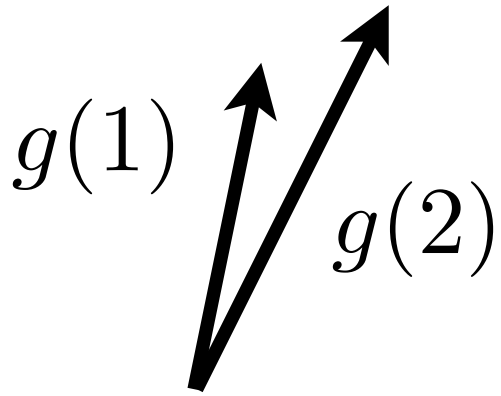
SOFTMAX EXTENSION

Multilinear: $F(\mathbf{x}) = \sum_Y \prod_{i \in Y} x_i \prod_{i \notin Y} (1 - x_i) \log f(Y)$

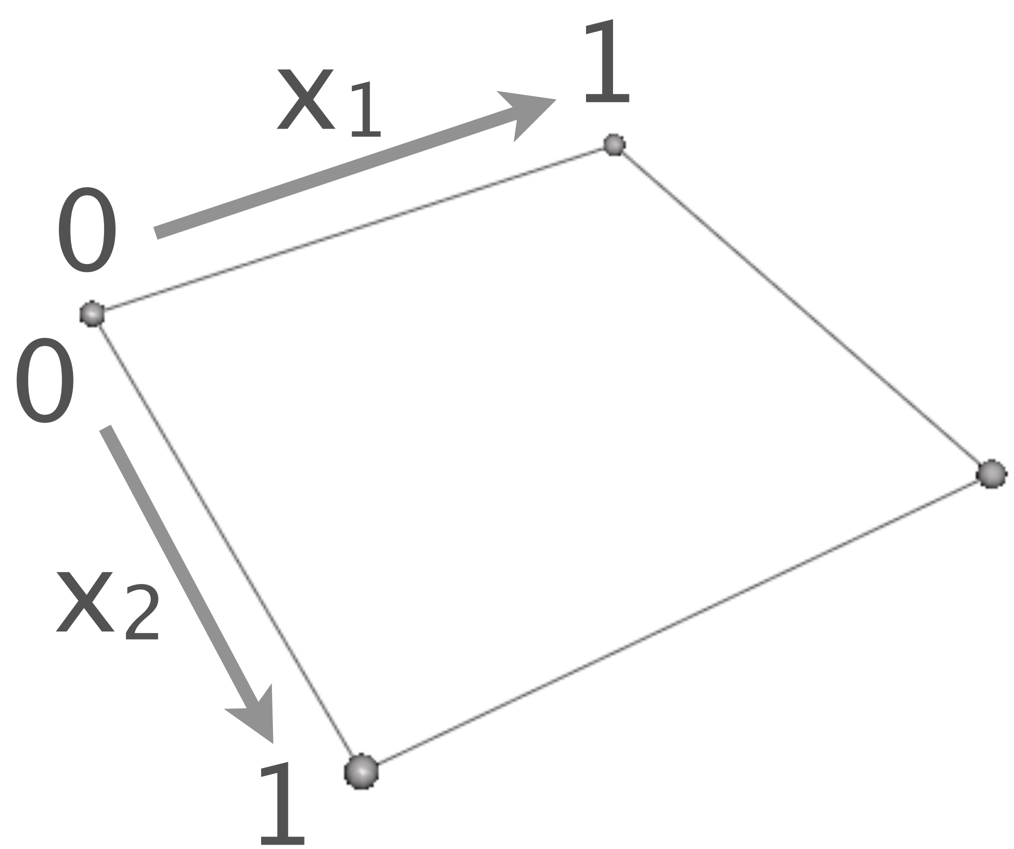
Softmax: $\tilde{F}(\mathbf{x}) = \log \sum_Y \prod_{i \in Y} x_i \prod_{i \notin Y} (1 - x_i) f(Y)$

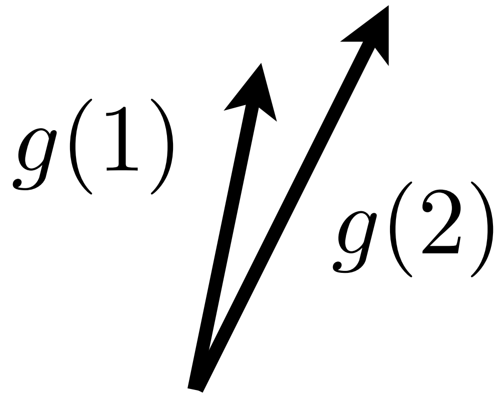
$N = 2$





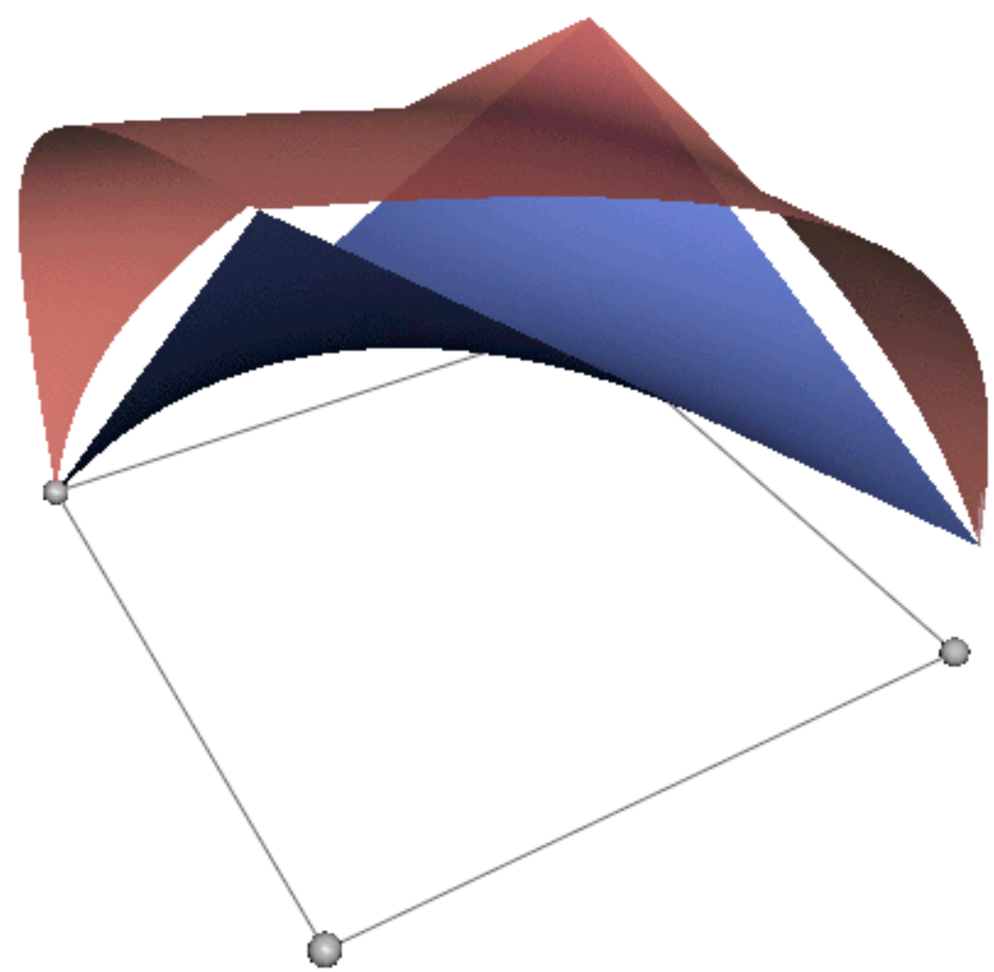
Relaxed domain

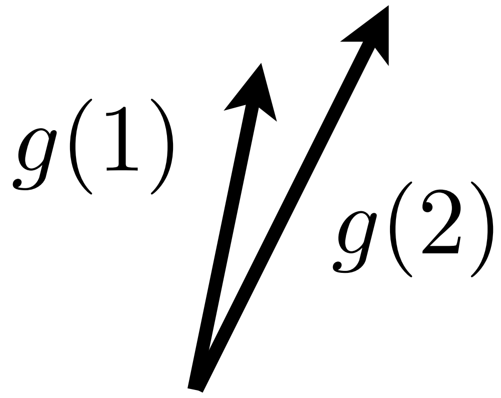




Softmax extension

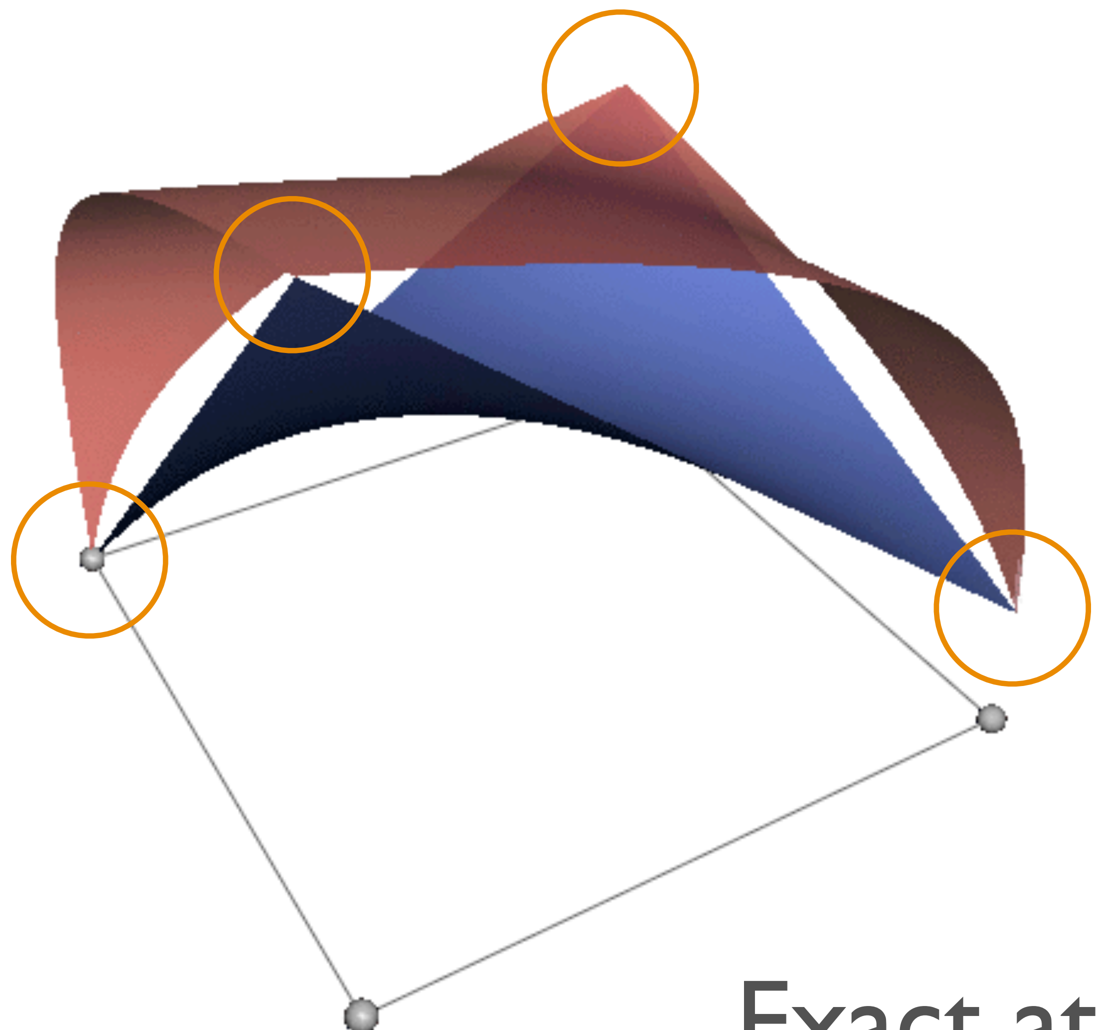
Multilinear extension



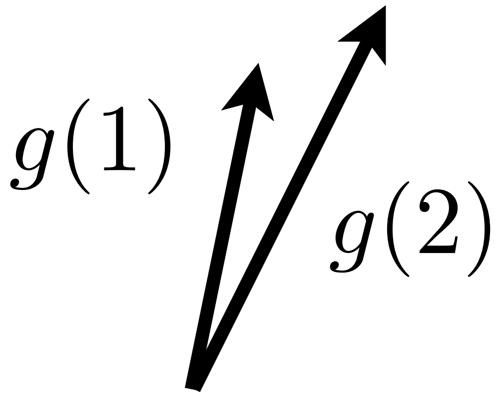


Softmax extension

Multilinear extension

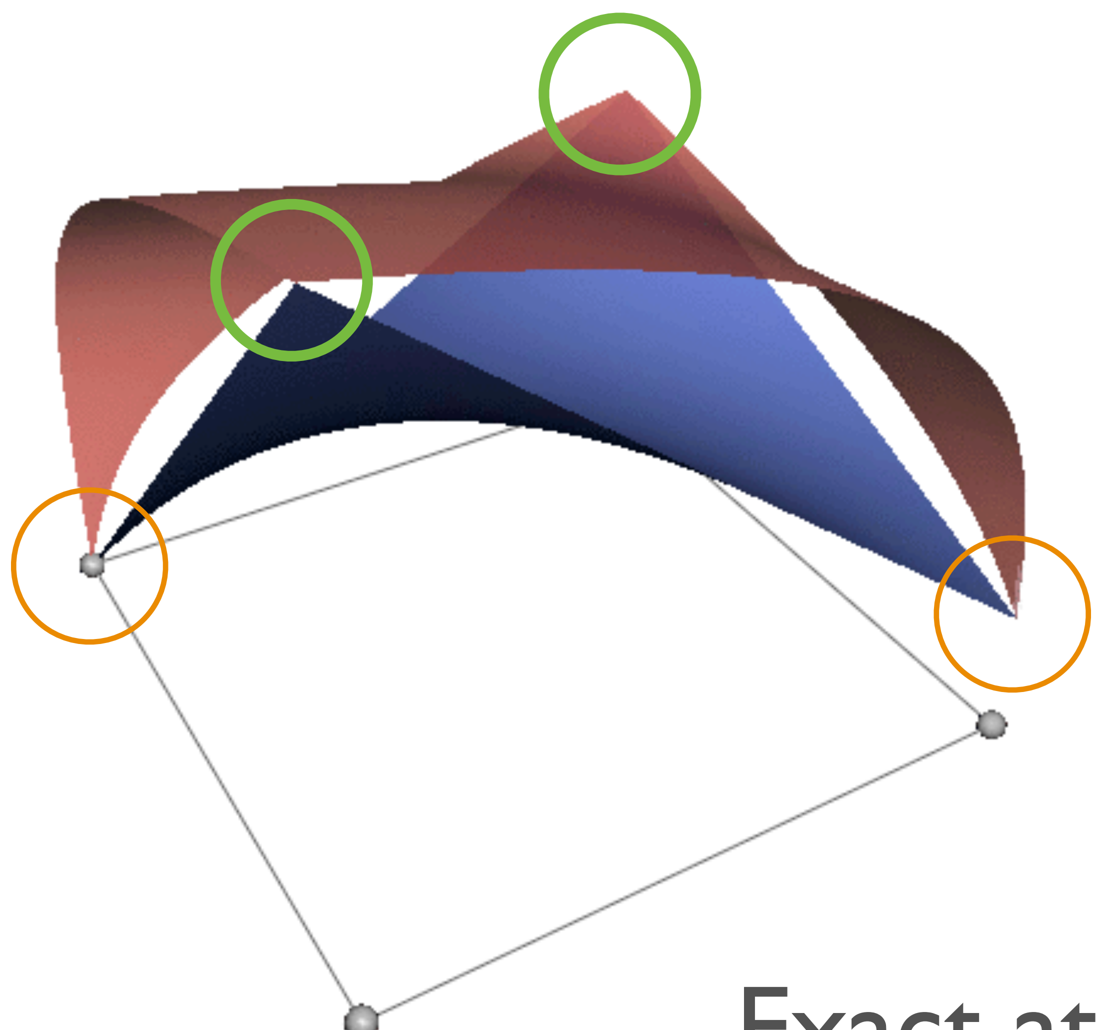


Exact at integral points



Softmax extension

Multilinear extension



Exact at integral points

SOFTMAX EXTENSION

$$\tilde{F}(\mathbf{x}) = \log \sum_Y \prod_{i \in Y} x_i \prod_{i \notin Y} (1 - x_i) f(Y)$$

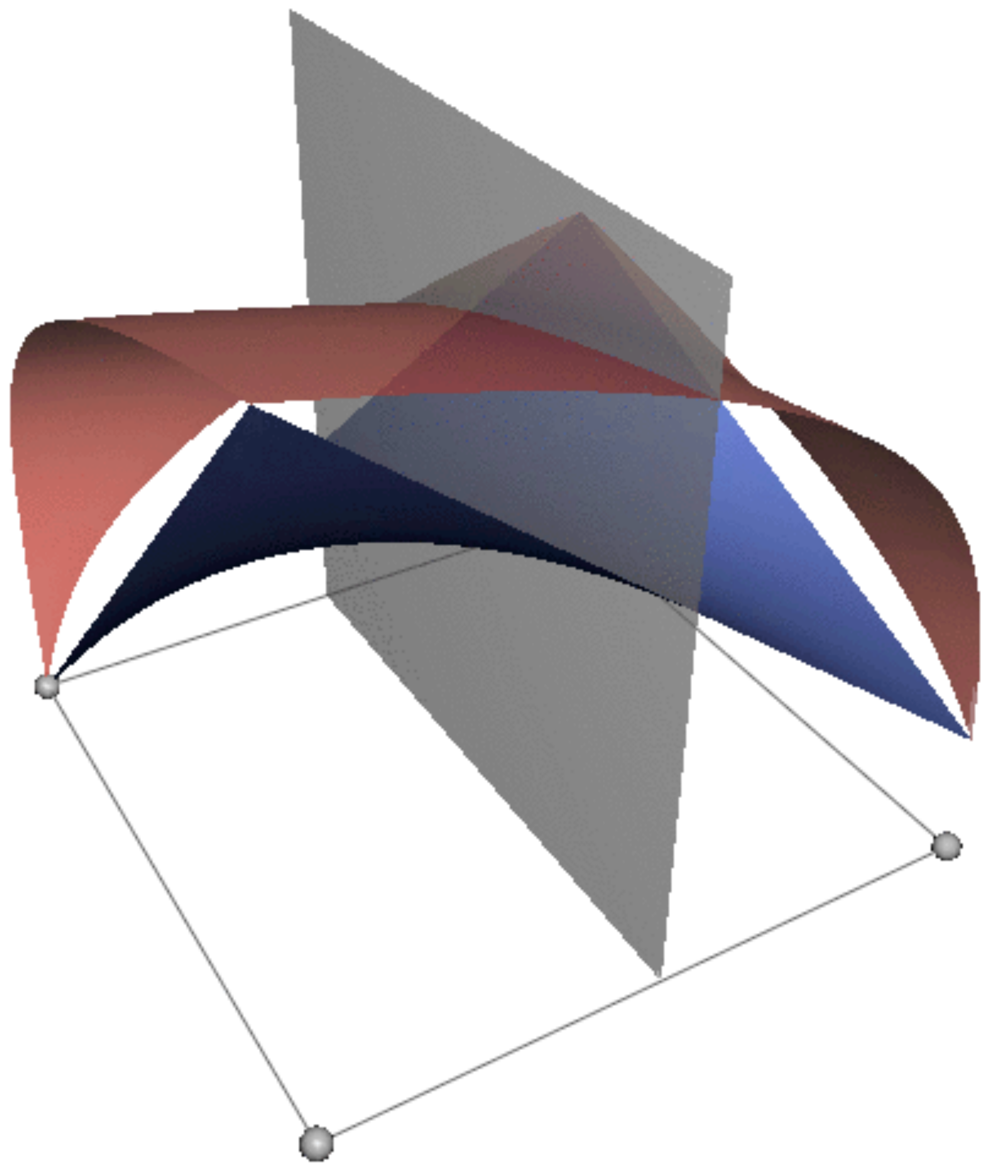
Theorem:

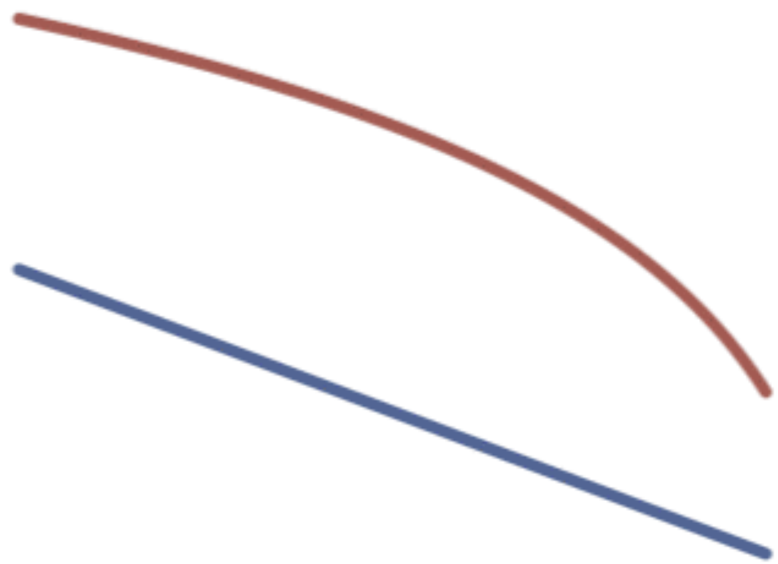
Efficiently computable for $f(Y) = \det(L_Y)$

$$O(N^3)$$

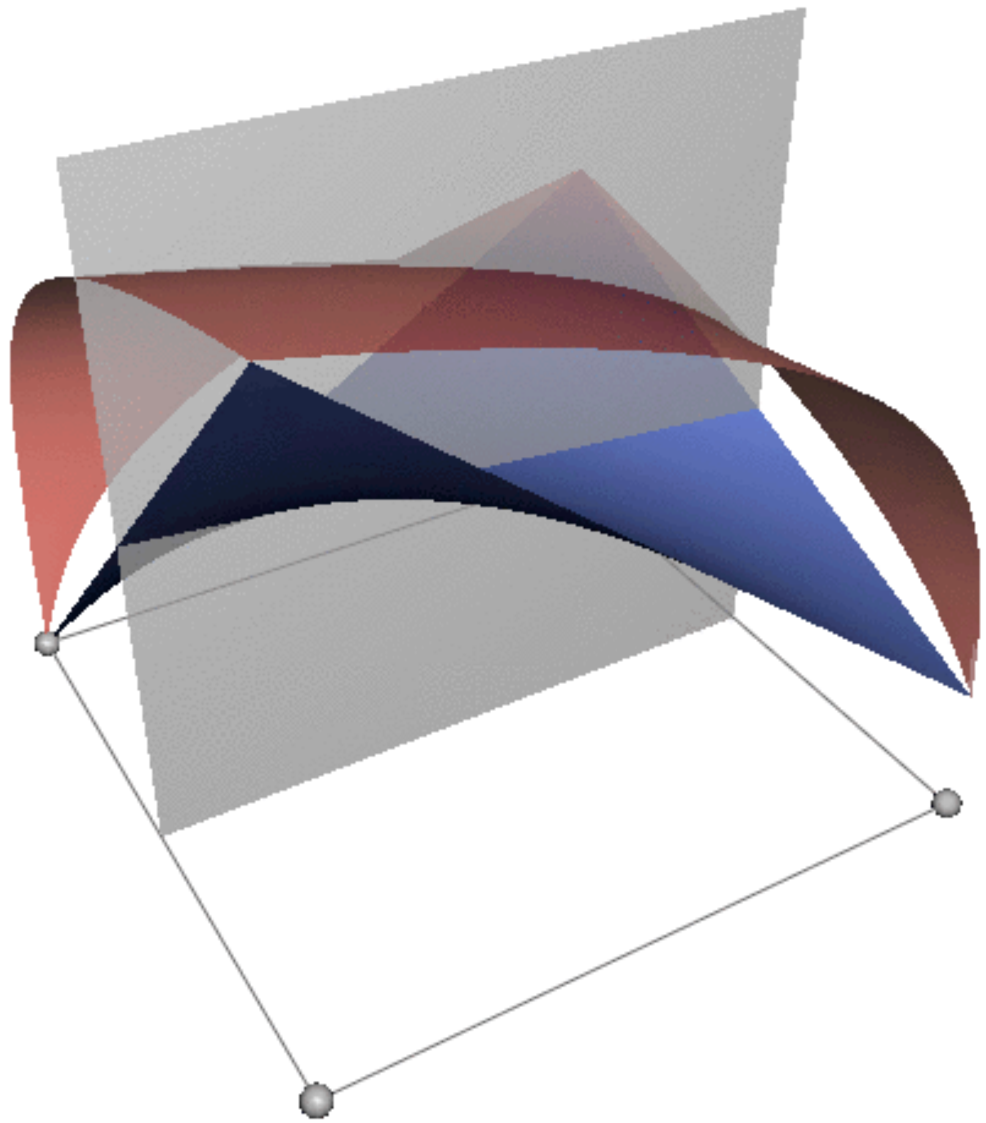
$$\tilde{F}(\mathbf{x}) = \det(\text{diag}(\mathbf{x})(L - I) + I)$$

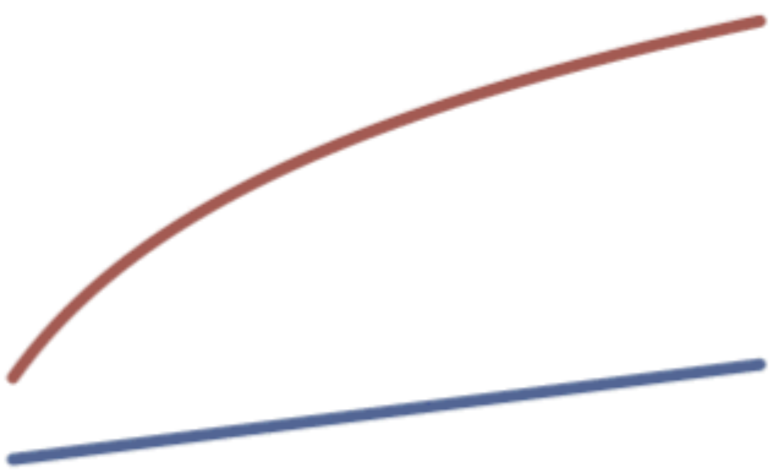
Concave in all-positive/all-negative directions





Not necessarily concave in other directions





APPROXIMATION GUARANTEE

Theorem: Concavity in all-positive directions

+ Submodularity $\Longrightarrow$

$$\text{LOCAL OPT of } \tilde{F} \geq \frac{1}{4} \max_{\mathbf{x}} \tilde{F}(\mathbf{x}) \geq \frac{1}{4} \max_{Y} \log\det(L_Y)$$

Theorem: In the unconstrained case, LOCAL OPT will be integer (no rounding necessary).

Constrained: No guarantees, but in practice pipage rounding and thresholding work well.

$\max_{Y \in S}$

BASELINE

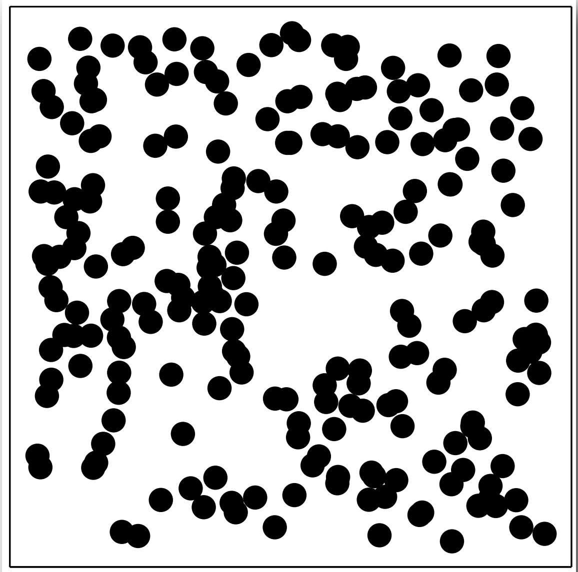
Monotone:
"greedy" (1 - 1/e)-approx
Nemhauser and Wolsey (1978)

Basic idea: Start from Y = {} and find the single item to add that most increases the score. Iterate until no remaining item increases the score.

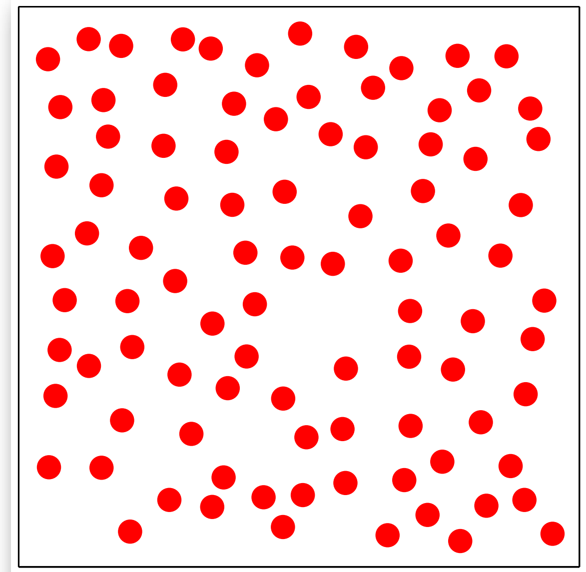
SYNTHETIC EXPERIMENTS

Unconstrained

$\max_Y$



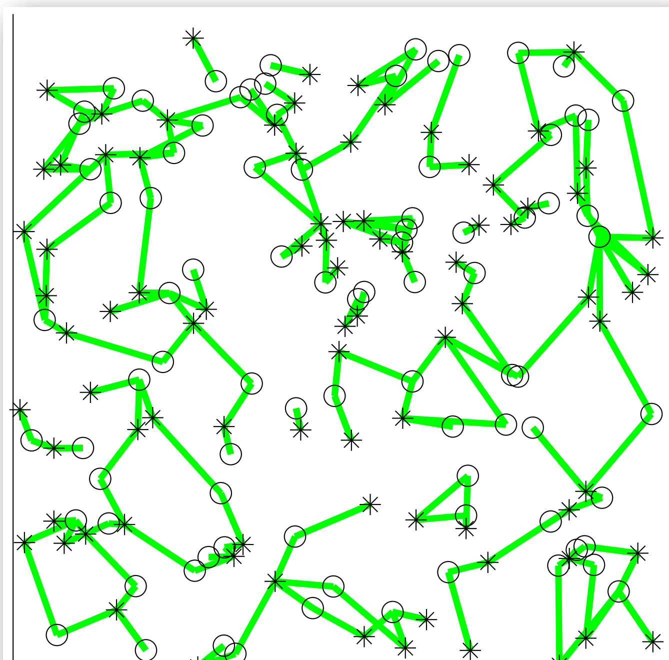
All points



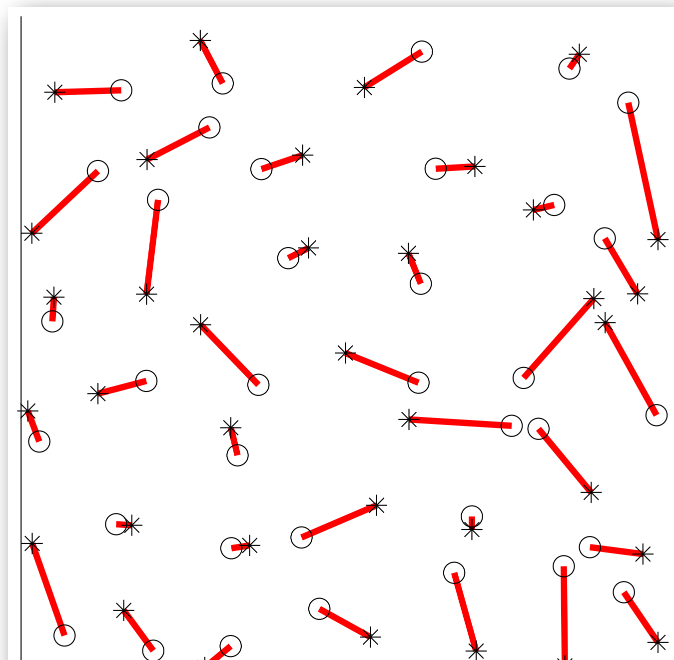
DPP (approx) MAP

Constrained

$\max_{Y \in S}$

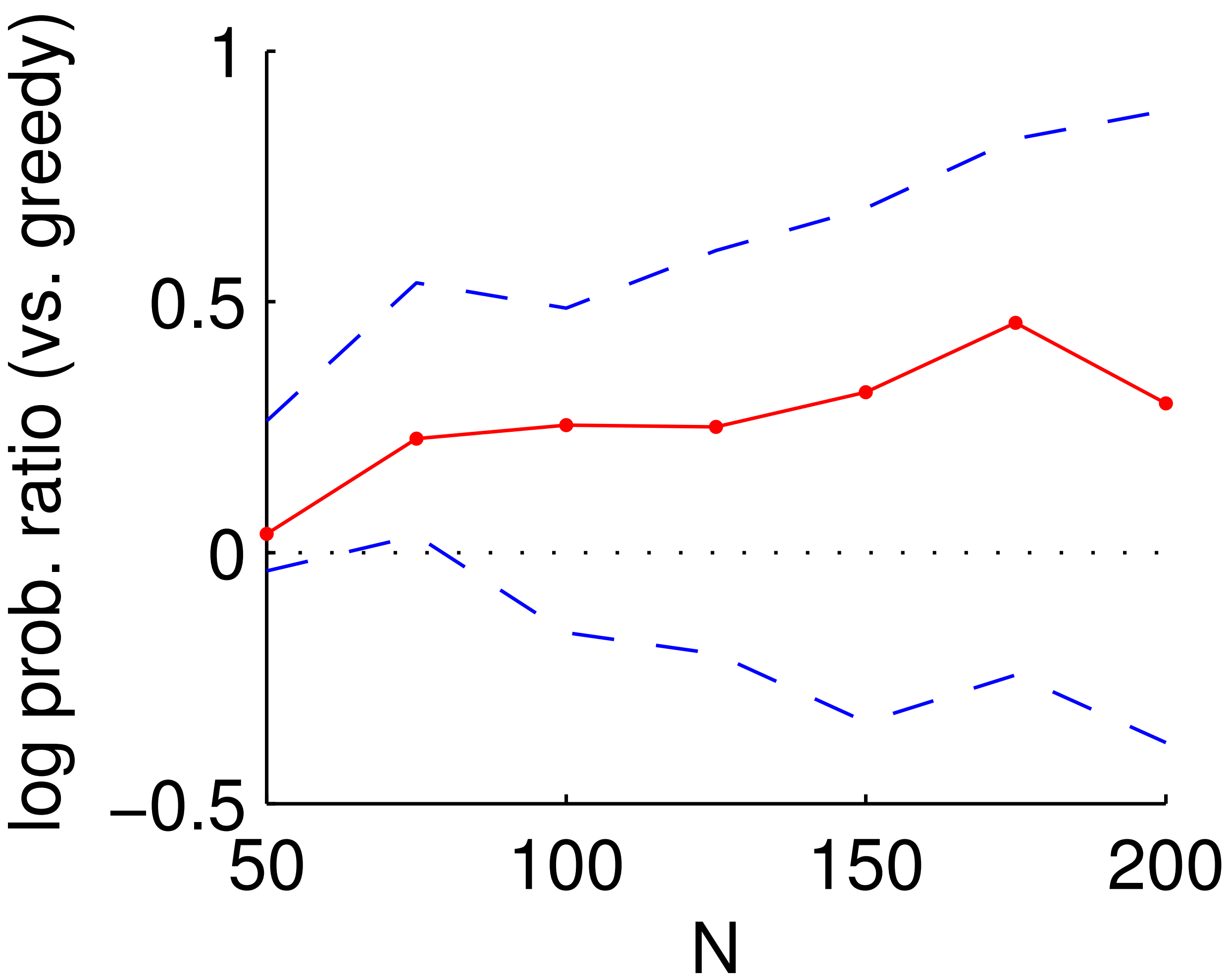


Many matches



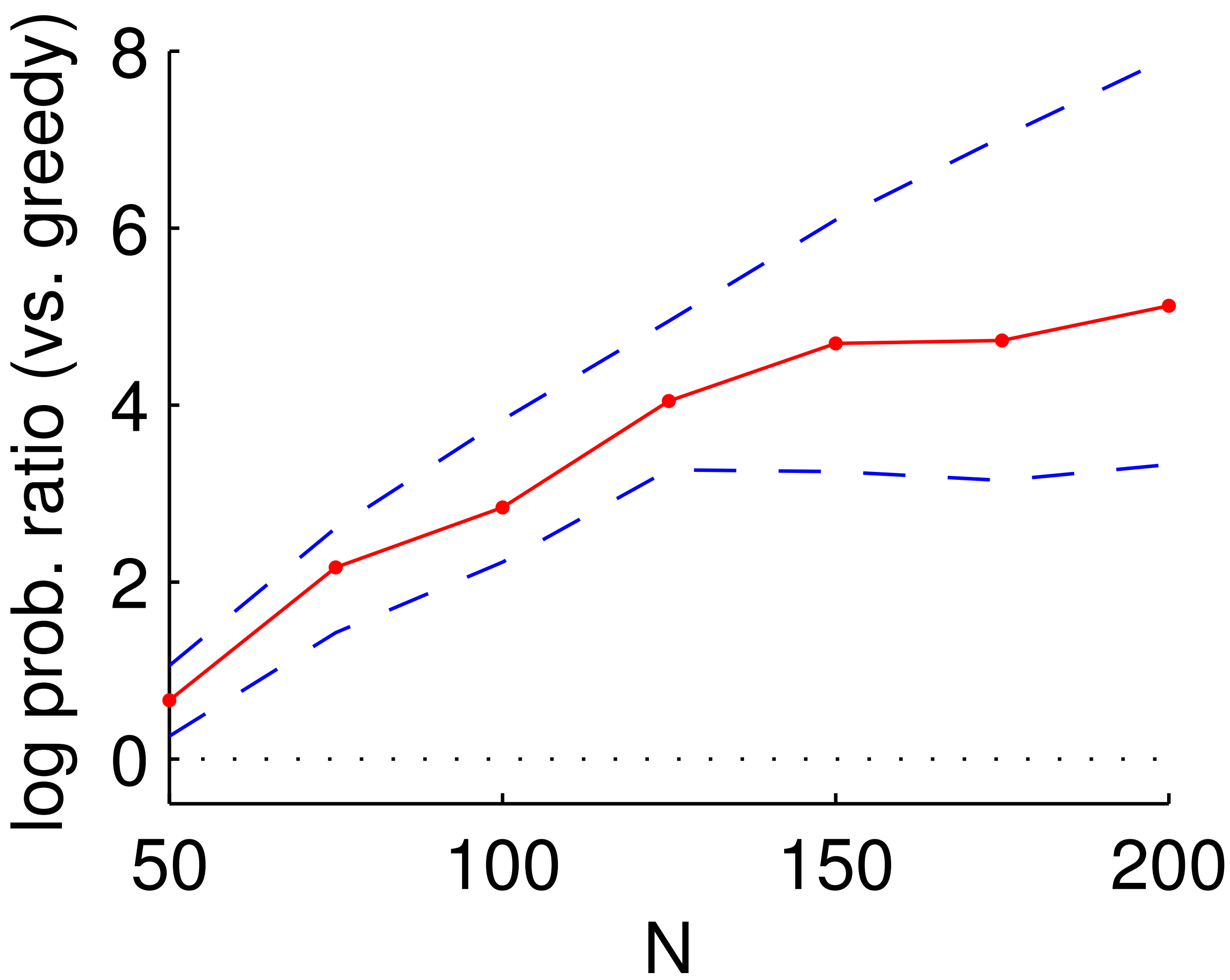
DPP MAP 1:1 matching

EFFECTIVENESS EVAL



Unconstrained

$\max_{Y}$



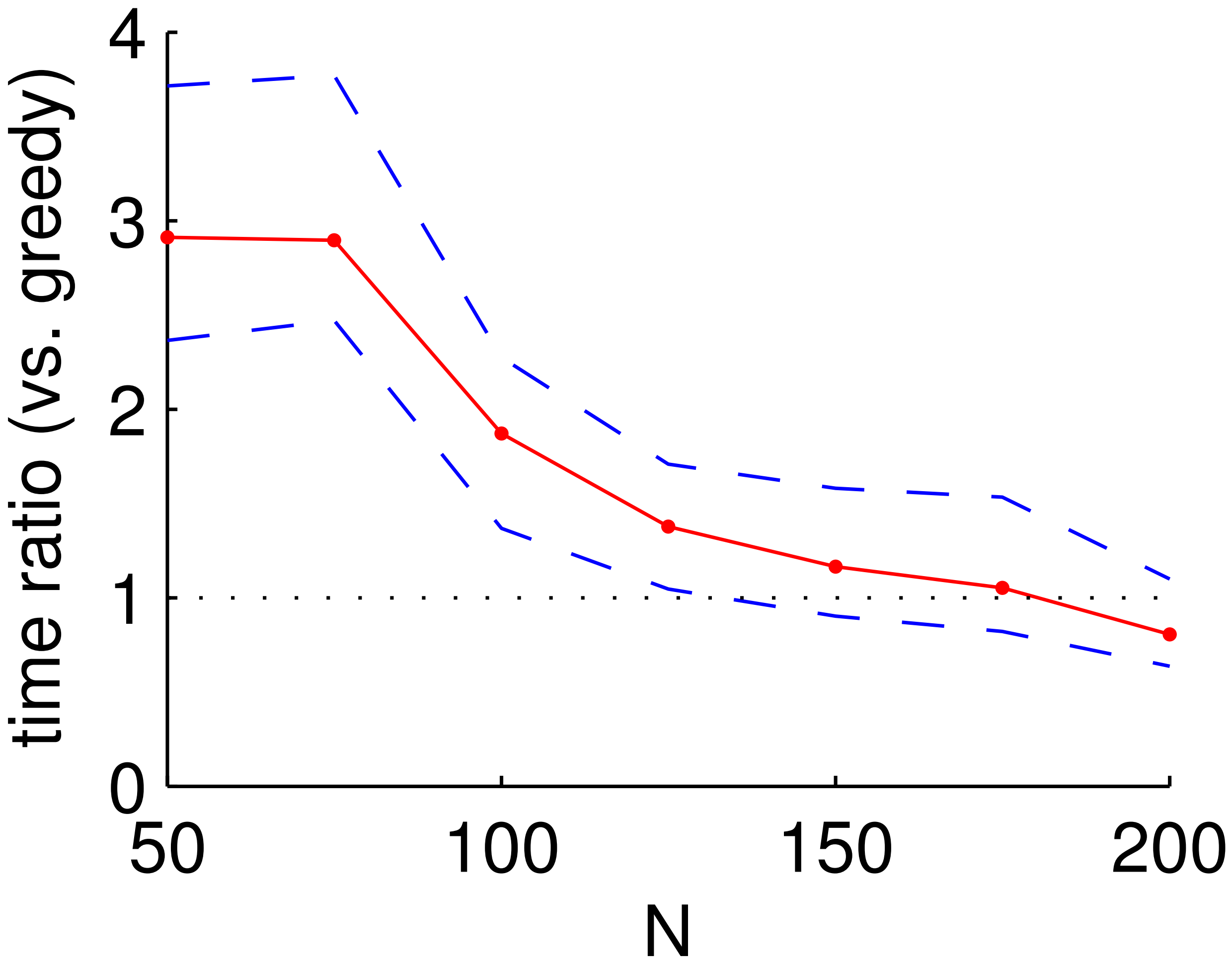
Constrained

$\max_{Y \in S}$

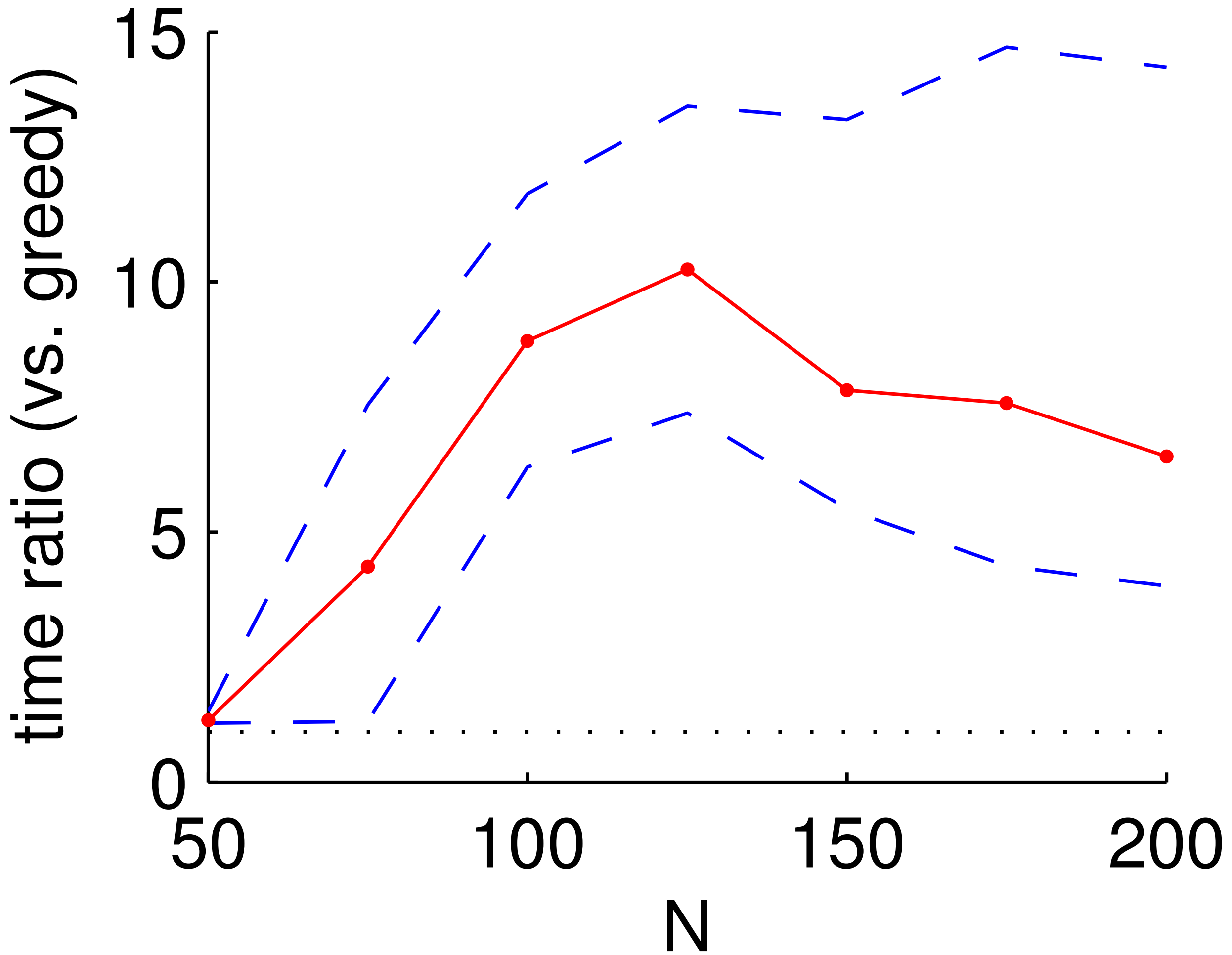
EFFICIENCY EVAL

Naive greedy algorithm: $O(N^5)$
Optimized version: $O(N^4)$

EFFICIENCY EVAL



Unconstrained
$\max_Y$



Constrained
$\max_{Y \in S}$

MATCHED SUMMARIZATION

20 Republican primary debates







Average of 179 quotes per candidate

MATCHED SUMMARIZATION

bowl
base
friend
nation
middle-class
obama
reduct
dividend
peopl
except
gain
time
interest
way
inch
term
place
amount
save
american
incom
bowles-simpson
govern
scale
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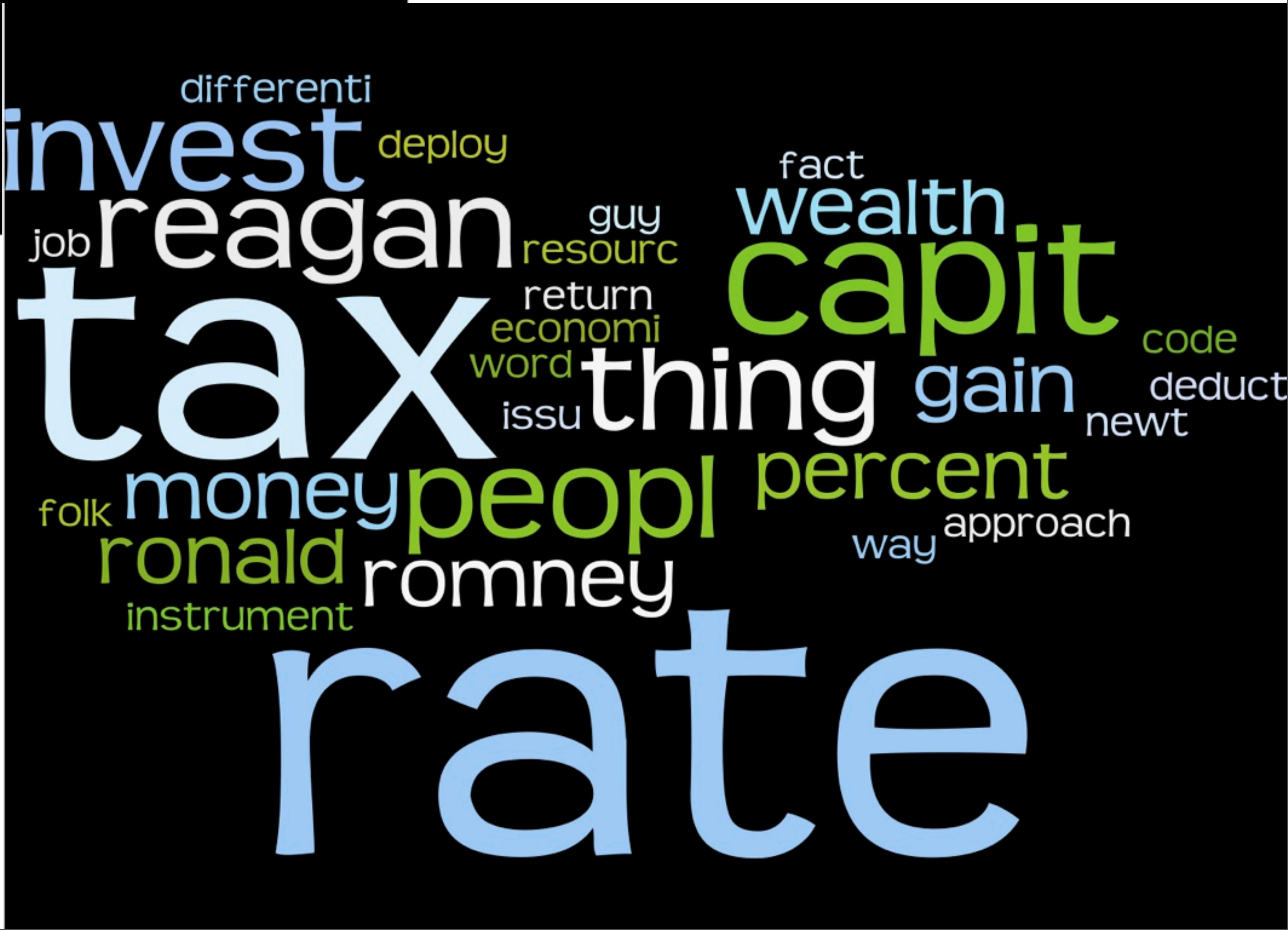
capit
exempt
simpson
incent
democrat
percent
tax



differenti
invest
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fact
wealth
job
reagan
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- **R1 (taxes): No tax on interest, dividends, or capital gains.**
- **R2 (law): We're not going to have Sharia law applied in U.S. courts.**
- **R3 (healthcare): I will ... grant a waiver from Obamacare to all 50 states.**
- **R4 (aid): We're spending more on foreign aid than we ought to.**
- **R5(healthcare): If you think what we did in Massachusetts and what President Obama did are the same, boy, take a closer look.**

- **S1 (taxes): I don't believe in a zero capital gains tax rate.**
- **S2 (taxes): Manufacture in America, you aren't going to pay any taxes.**
- **S3 (aid): Zeroing out foreign aid ... that's absolutely the wrong course.**
- **S4 (ethanol): I voted against ethanol subsidies my entire time in Congress.**
- **S5 (healthcare): Obamacare ... is going to blow a hole in the budget.**

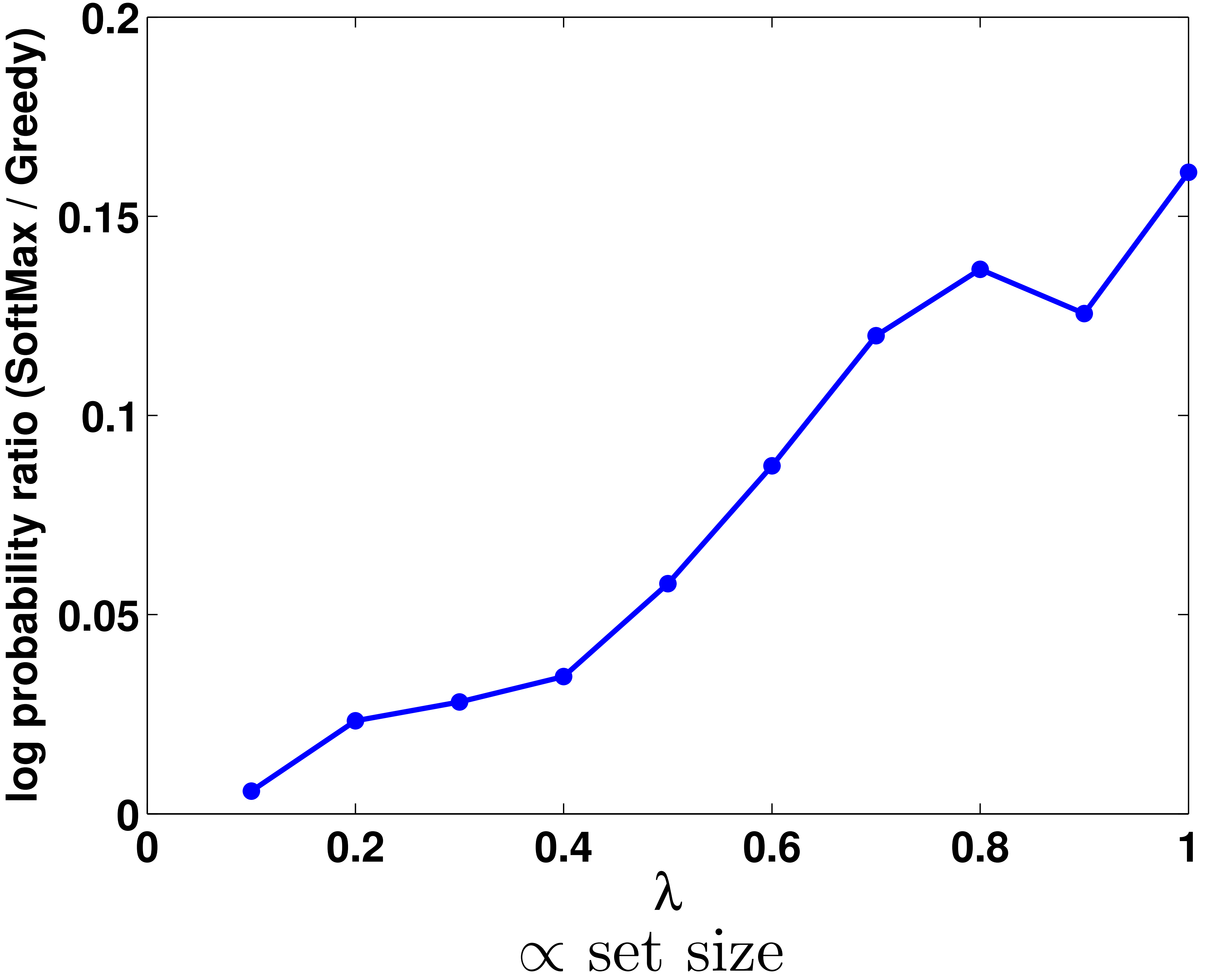
Matched summary

- **R1 (taxes): No tax on interest, dividends, or capital gains.**
- **S1 (taxes): I don’t believe in a zero capital gains tax rate.**

- **R3 (healthcare): I will … grant a waiver from Obamacare to all 50 states.**
- **S5 (healthcare): Obamacare … is going to blow a hole in the budget.**

- **R4 (aid): We’re spending more on foreign aid than we ought to.**
- **S3 (aid): Zeroing out foreign aid … that’s absolutely the wrong course.**

PERFORMANCE



SUMMARY

Efficient, effective approximate DPP MAP algorithm for subset selection problems

Code + data:

http://www.seas.upenn.edu/~jengi/dpp-map.html

SUMMARY

Code + data:

http://www.seas.upenn.edu/~jengi/dpp-map.html

Future work:

- Other applications:
  - sensor selection
    [A. Krause, A. Singh, and C. Guestrin. Near-Optimal Sensor Placements in Gaussian Processes, 2008.]
  - text summarization
    [H. Lin and J. Bilmes. Multi-Document Summarization via Budgeted Maximization of Submodular Functions, 2010.]
- Other submodular functions for which the softmax extension is efficiently computable?

Poster W35