Tutorial 3 for ERG2040C

Zichao Yang
zcyang@cse.cuhk.edu.hk

Outline

- Review
  - Axioms of probability
- Examples
  - Events with different weights (problem last time)
  - Register
  - Sample with replacement
  - Cube coloring
  - Network capacity

Review--axioms of probability

- Sample space S
  - The set of all possible outcomes of experiments, every element of the set is an outcome
- Event E
  - A subset of sample space, $E \subset S$
- Axioms of probability
  - $0 \leq P(E) \leq 1$
  - $P(S) = 1$
  - For mutually exclusive events, $P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$
- If S is finite and each one point set is assumed to have equal probability, then $P(E) = \frac{|E|}{|S|}$

Example 1: events with different weights

- Here are two teams of ordered players
  - Players A1, A2, A3, A4, A5 from Team A
  - Players B1, B2, B3, B4, B5 from Team B
  - Each player is equally likely to win or lose against player
- First round: A1 vs. B1
- Nth round:
  - The loser of the previous round is out
  - The winner of the previous round vs. the next player of the loser's team
- The play ends when all players in one team are out
- What's the probability that team A wins and four players in A are out?

Solution1:

- Use a sequence of player to represent an outcome
  - The sequence is (from left side to right side)
    - The loser in each round (from left side to right side)
    - When the play ends, put the winner and those who do not participate in the sequence(because their previous teammates killed all the competitors)
  - For example: (red--losers, green--last winner, purple--stander- by)
    - A1,A2,A3,A4,A5,B1,B2,B3,B4,B5 means that B1 wins against A1 to A5
    - A1,A2,A3,A4,B1,B2,B3,B4,A5,B5 means team B wins while B5 is the last winner and four of B are out
    - The relative order of A1 to A5, and B1 to B5 do not change

Solution1:

- Sample space:
  - For every sequence, we can find the corresponding result
  - Every sequence corresponds to different result
  - So the sample space is the all the possible sequences, it is equal to choose 5 positions from 10, $\binom{10}{5}$
- What is the event:
  - Team A wins and four members are out
  - X,X,X,X,X,X,X,X,B5,A5
  - A5 is the last because A5 is the last winner
  - B5 is the penultimate because four members of A are out
  - Number of sequence $\binom{8}{4}$
- Probability ???

$$\binom{8}{4} / \binom{10}{5}$$

WRONG!!!

Solution 1:

- How many rounds of play?—how many losers
  - A1,A2,A3,A4,A5,B1,B2,B3,B4,B5: **5 rounds,** probability: $\frac{1}{2^5}$
  - A1,A2,A3,A4,B1,B2,B3,B4,A5,B5: **9 rounds,** probability: $\frac{1}{2^9}$
  - Different sequences have different probability (weight)
- Consider the event:
  - For every sequence, 9 rounds(A have 4 losers and B has 5 losers)
  - How many sequences: $\binom{8}{4}$
  - So the probability is: $\frac{\binom{8}{4}}{2^9}$

Remark

- When the sample points are of different weights, we can not use division to calculate the probability
- In many cases, the weight of sample point is not equal, this will happen in the following examples

Example 2: registers

- A register contains 8 random binary digits which are mutually independent. Each digits is a zero of a one with equal probability. Calculate the probability of following event:
  - E1: no two neighboring digits are the same
  - E2: some cyclic shift of the register contents is equal to 01100110
  - E3: the register contains exactly four zeros
  - E4: there is a run of at least six consecutive ones

Solution 2:

- Sample space S:
  - $\{x_1x_2x_3x_4x_5x_6x_7x_8 : x_i \in \{0,1\} \text{ for each i}\}$ the total number is 256=2^8
- Different events
  - E1={01010101, 10101010} and P(E1)=2/256
  - E2={00110011, 01100110, 11001100, 10011001} and P(E2)=4/256
  - E3= $\{x : x_1 + \ldots + x_8 = 4\}$ and P(E3)= $\frac{\binom{8}{4}}{256}$
  - E4={11111111, 11111110, 11111101, 10111111, 01111111, 00111111, 01111110, 11111100} and P(E4)=8/256

Example 3: sample with replacement

- An urn contains n white and m black balls, where n and m are positive numbers
  - a) If two balls are randomly withdrawn, what is the probability that they are of the same color?
  - b) If a ball is randomly withdrawn and when replaced before the second one is drawn, what is the probability that the withdrawn balls are the same color?
  - c) Compare the probability which is bigger.

Solution3:

- a)
  - If the color is white, then probability is:

$$\frac{\binom{n}{2}}{\binom{n+m}{2}} = \frac{n(n-1)}{(n+m)(n+m-1)}$$

  - If the color is black, the probability is :

$$\frac{\binom{m}{2}}{\binom{n+m}{2}} = \frac{m(m-1)}{(n+m)(n+m-1)}$$

  - The total probability is:

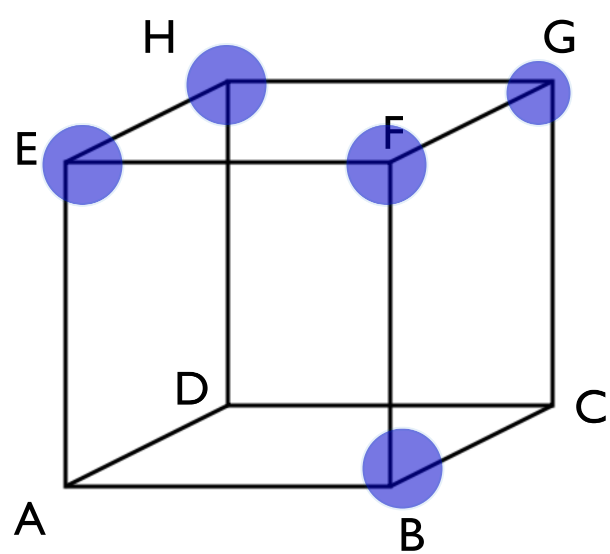
$$\frac{n(n-1)+m(m-1)}{(n+m)(n+m-1)}$$

Solution 3:

- b)
  - Since the picked ball is replaced, the probability of choosing a while ball or a back ball stay unchanged
  - Two while balls: $\left(\frac{n}{n+m}\right)^2$
  - Two black balls: $\left(\frac{m}{n+m}\right)^2$
  - The total probability is : $\frac{n^2+m^2}{(n+m)^2}$
- c) $\frac{n(n-1)+m(m-1)}{(n+m)(n+m-1)} < \frac{n^2+m^2}{(n+m)^2}$
  - The probability in the putting back case is bigger!

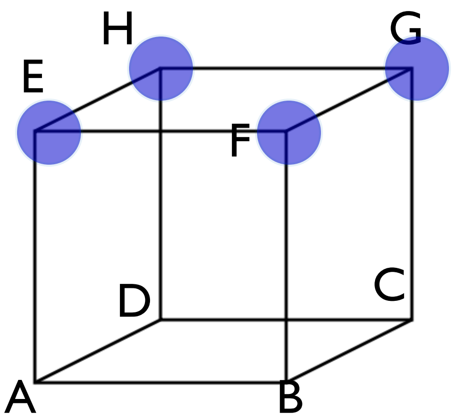
Example 4: cube coloring

- Suppose each corner of a cube is colored blue with probability p, red with probability $1-p$. Let E denote the event that at least one face of the cube has all four corners colored blue.
  - Find P[E]



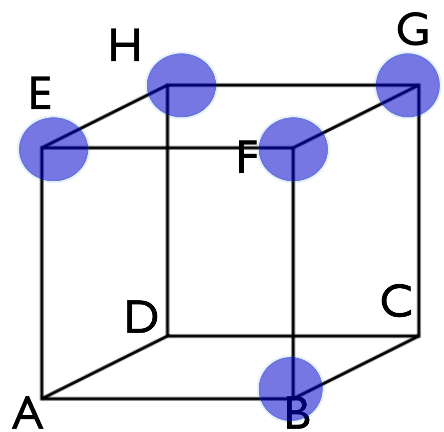
Solution 4:

- i=4



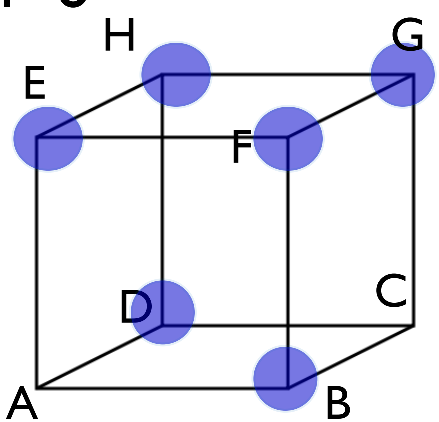
Appear on different faces: 6

i=5

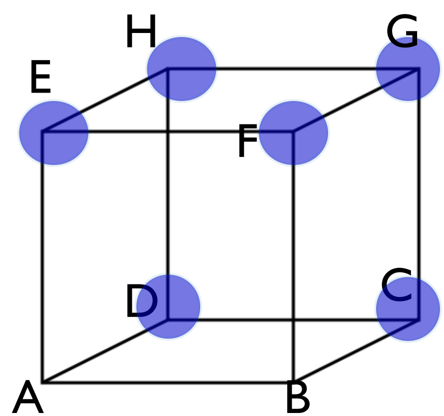


4*6=24

- i=6



Diagonal: 2*6=12



4*6/2=12

Solution 4:

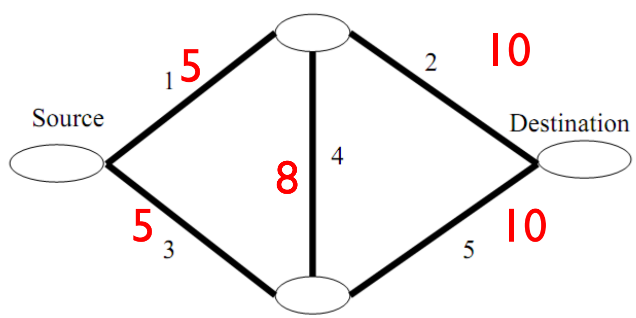
- The number of blue corners must be greater than 4
- Denote E_i as the event that there are exactly i corners colored with blue and at least one face of the cube has four corners colored blue
- Then $P(E) = \sum_{i=4}^{8} P(E_i)$

- $P(E_4) = 6p^4(1-p)^4$
- $P(E_5) = 24p^5(1-p)^3$
- $P(E_6) = 24p^6(1-p)^2$
- $P(E_7) = 8p^7(1-p)^1$

  $P(E_8) = p^8$

Example 5:

- A communication network is shown. The link capacities in megabits per second(Mbps) are given by $C_1 = C_3 = 5$, and $C_2 = C_5 = 10$, $C_4 = 8$, and are the same in each direction. Information flow from the source to the destination can be split among multiple paths. Each link fails with probability p independently. Let X be defined as the maximum rate (in Mbits per second) at which data can be sent from the source node to the destination node. Find $P(X \geq 8)$



Solution 5:

- If all links are working, then the maximum communication rate is 10 Mbps
- If X>=8, then
  - Link 1 and link 3 must be working
  - Then we have the following conditions that X>=8
  - 1,3, 2, 4, 5 work, the probability is $(1-p)^5$
  - 1, 3, 2, 4 work, 5 fails, the probability is $(1-p)^4 p$
  - 1, 3, 2, 5 work, 4 fails, the probability is $(1-p)^4 p$
  - 1, 3, 5, 4 work, 2 fails, the probability is $(1-p)^4 p$
  - So $P(X \geq 8) = (1-p)^2((1-p)^3 + 3(1-p)^2 p)$
- What about $P(X \geq 5)$?