Notes 16: Effective resistance

As in the last lecture, let $H = (V, E)$ be a connected, undirected graph (representing an electrical network) with positive edge weights $w : E \to \mathbb{R}_+$.

The goal of this lecture is to develop tools for fast algorithms to approximately solve Laplace equations.

1. Effective resistance

Given any nodes a and b, we can treat the whole electrical network H as a single resistor between a and b. What is the resistance of this resistor?

If we inject one unit of external current at a and remove one unit of current at b, we can measure the resulting potential difference $v(a) - v(b)$. Ohm's law tells us to expect

$$v(a) - v(b) = i(a, b)R_{\text{eff}}(a, b) .$$

Thus, we define the effective resistance $R_{\text{eff}}(a, b)$ between a and b so that this equation holds.

This corresponds to the external current vector $u = \mathbb{1}_a - \mathbb{1}_b$. The above discussion implies the voltage vector due to u is $v = L^+u$. The potential difference $v(a) - v(b)$, and hence $R_{\text{eff}}(a, b)$, is $(\mathbb{1}_a - \mathbb{1}_b)^\top L^+(\mathbb{1}_a - \mathbb{1}_b)$.

Since L is positive semidefinite, so is L^+, and therefore it has a square-root $L^{+/2}$. In terms of spectral decomposition using nonnegative eigenvalues λ_ℓ and orthonormal eigenvectors ψ_ℓ,

$$L = \sum_\ell \lambda_\ell \psi_\ell \psi_\ell^\top \qquad \Longrightarrow \qquad L^{+/2} = \sum_{\ell\colon \lambda_\ell > 0} \frac{1}{\sqrt{\lambda_\ell}} \psi_\ell \psi_\ell^\top .$$

Therefore

$$R_{\text{eff}}(a, b) = (\mathbb{1}_a - \mathbb{1}_b)^\top L^+(\mathbb{1}_a - \mathbb{1}_b) = (\mathbb{1}_a - \mathbb{1}_b)^\top (L^{+/2})^\top L^{+/2}(\mathbb{1}_a - \mathbb{1}_b) = \|L^{+/2}\mathbb{1}_a - L^{+/2}\mathbb{1}_b\|_2^2 .$$

In other words, if we represent every node a as the vector $L^{+/2}\mathbb{1}_a$, then $R_{\text{eff}}(a, b)$ is the squared Euclidean distance between the corresponding vectors $L^{+/2}\mathbb{1}_a$ and $L^{+/2}\mathbb{1}_b$. This map $a \mapsto L^{+/2}\mathbb{1}_a$ is sometimes called the effective resistance embedding.

2. Equivalent networks, Gaussian elimination

We just considered what happens when two nodes are under external influence — the rest of the network can be represented as a single resistor. We now do the same when a subset $B \subseteq V$ of nodes are under external influence.

We call B the set of boundary nodes and $I = V \setminus B$ the set of internal nodes. You may imagine that we can attach electrodes of batteries to nodes in B but not in I. So we can set voltages of nodes in V, while voltages of nodes in I are determined by electrical flow of the batteries.

When $B = V$, the Laplace operator L maps voltage vector $v \in \mathbb{R}^B$ to vector of external currents $u \in \mathbb{R}^B$. Now for a general subset $B \subseteq V$, we want to find a matrix L_B such that

$$u_B = L_B v_B .$$

Turns out L_B is a Laplacian matrix (easy exercise), and is obtained by applying Gaussian elimination to remove the internal nodes.

To be concrete, we take $V = \{1, \ldots, n\}$, $B = \{2, \ldots, n\}$, and we eliminate the internal node 1 using Gaussian elimination. Given any voltage vector $v_B \in \mathbb{R}^B$, we want to find $v \in \mathbb{R}^V$ such that $v(b) = v_B(b)$ for every $b \in B$, and

$$0 = u(1) = \sum_{b\sim 1} i(1, b) = \sum_{b\sim 1} w(1, b)(v(1) - v(b)) .$$

Rearranging,

$$v(1) = \frac{1}{d(1)} \sum_{b\sim 1} w(1, b)v(b) .$$

This means $v(1)$ is a weighted average of voltages of its neighbors b. It also means when solving the Laplace equation $u = Lv$, we will substitute $v(1)$ as the right-hand-side whenever $v(1)$ appears. The term $v(1)$ only appears in the equation for $u(a)$ when a is a neighbor of 1, and the equation is

$$u(a) = d(a)v(a) - \sum_{b\sim a} w(a,b)v(b) \ .$$

After substituting $v(1)$, the equation for $u(a)$ becomes

$$u(a) = d(a)v(a) - \sum_{b\sim a,\ b\neq 1} w(a,b)v(b) - \frac{w(1,a)}{d(1)} \sum_{b\sim 1} w(1,b)v(b) \ .$$

One of the term in the last sum is in fact node a, so the equation should be rewritten as

$$\begin{aligned} u(a) &= d(a)v(a) - \sum_{b\sim a,\ b\neq 1} w(a,b)v(b) - \frac{w(1,a)}{d(1)} \sum_{b\sim 1, b\neq a} w(1,b)v(b) - \frac{w(1,a)^2}{d(1)} v(a) \\ &= \left(d(a) - \frac{w(1,a)^2}{d(1)} \right) v(a) - \sum_{b\sim a,\ b\neq 1} w(a,b)v(b) - \frac{w(1,a)}{d(1)} \sum_{b\sim 1, b\neq a} w(1,b)v(b) \ . \end{aligned}$$

This is exactly the result of applying Gaussian elimination to eliminate the variable $v(1)$ using the equation $u(1) = 0$.

3. Distance

A distance d (also known as a metric) is any real-valued function on pair of vertices such that

- (Nonnegativity) $d(a,b) \geqslant 0$ for any vertices a and b
- (Identity of indiscernibles) $d(a,b) = 0$ if and only if $a = b$
- (Symmetry) $d(a,b) = d(b,a)$ for any a and b
- (Triangle inequality/subadditivity) $d(a,c) \leqslant d(a,b) + d(b,c)$ for any a, b and c

We now argue that effective resistance R_{eff} is a distance. The first three properties easily follow from §1 of this notes. It remains to prove the last property (triangle inequality).

We need the following simple observation: Given a unit electrical flow from a to b, the corresponding voltage vector $v \in \mathbb{R}^V$ satisfies $v(a) \geqslant v(c) \geqslant v(b)$ for any node c.

This observation holds because the voltage of any internal node c is a weighted average of its neighbors. To formally prove it, one can first consider the equivalent network with boundary $B = \{a, b, c\}$. The voltage of c in this equivalent network, after $v(a)$ and $v(b)$ are fixed, will be a weighted average of $v(a)$ and $v(b)$, and hence between them.

Proposition 3.1. $R_{\text{eff}}(a,c) \leqslant R_{\text{eff}}(a,b) + R_{\text{eff}}(b,c)$.

Proof. Let $u_{a,b} = \mathbb{1}_a - \mathbb{1}_b$ be the external current for the unit current flow from a to b. Similarly, $u_{b,c} = \mathbb{1}_b - \mathbb{1}_c$ and $u_{a,c} = \mathbb{1}_a - \mathbb{1}_c$. Note that

$$u_{a,c} = u_{a,b} + u_{b,c} \ .$$

Let $v_{a,b} = L^+ u_{a,b}$ be the voltage vector for $u_{a,b}$. Likewise $v_{b,c} = L^+ u_{b,c}$ and $v_{a,c} = L^+ u_{a,c}$. By linearity,

$$v_{a,c} = v_{a,b} + v_{b,c} \ ,$$

and

$$R_{\text{eff}}(a,c) = v_{a,c}(a) - v_{a,c}(c) = v_{a,b}(a) - v_{a,b}(c) + v_{b,c}(a) - v_{b,c}(c) \ .$$

By above observation, the first two terms

$$v_{a,b}(a) - v_{a,b}(c) \leqslant v_{a,b}(a) - v_{a,b}(b) = R_{\text{eff}}(a,b)$$

and similarly $v_{b,c}(a) - v_{b,c}(c) \leqslant v_{b,c}(b) - v_{b,c}(c) = R_{\text{eff}}(b,c)$. □

4. Equivalent electrical power

Effective resistance between a and b in a network is defined as the resistance of the equivalent resistor. Turns out the network and its equivalent resistor share more common properties than just the same resistance: they also dissipate the same power per unit flow.

Proof. The power dissipated per unit of a-b flow in the equivalent resistor is exactly $R_{\text{eff}}(a,b)$, due to Joule's law $P = I^2 R$.

The power dissipated in the network per unit of a-b flow is $i^\top W^{-1} i$, where W is the diagonal matrix of edge weights, and i is the unit electrical flow from a to b. Since i is induced by some voltage $v \in \mathbb{R}^V$ and $i = WBv$, the power dissipated is

$$i^\top W^{-1} i = (WBv)^\top W^{-1} (WBv) = v^\top B^\top WBv = v^\top Lv \ .$$

And since

$$R_{\text{eff}}(a,b) = (\mathbb{1}_a - \mathbb{1}_b)^\top L^+ (\mathbb{1}_a - \mathbb{1}_b) = (Lv)^\top L^+ (Lv) = v^\top Lv \ ,$$

the network dissipates the same power as the equivalent resistor.

In the last equation, the first equality relating effective resistance and L^+ is proved to §1 of this notes; the second equality is due to $Lv = \mathbb{1}_a - \mathbb{1}_b$ (that is, v is the voltage vector so that one unit of current flows from a to b); the last equality is $LL^+L = L$. □