CSCI5160 Approximation Algorithms — Spring 2020

Lecturer: Siu On Chan — *Scribe: Siu On Chan*

Notes 16: Electrical flow and Laplace equation

1. Resistor network and electrical flow

Consider a connected, undirected graph $H = (V, E)$ with positive edge weights $w : E \to \mathbb{R}_+$. We think of H as an electrical network where every edge $(a, b) \in E$ is a resistor with resistance $r(a, b) = 1/w(a, b)$. In other words, the electrical conductance of the resistor is $w(a, b)$; but we won't use this terminology to avoid confusion. We use letters a, b or c to denote nodes in H, because letters v or i will mean something else in an electrical network.

If we further attach a battery with positive electrode at node s and negative electrode at node t, electrical current will flow from source s to sink t along the wires (edges of H). We think of the battery as external to the electrical network H. The electrical flow can be specified by a function $i : \overrightarrow{E} \to \mathbb{R}$ that tells us the amount of electrical current along every directed edge $(a, b) \in \overrightarrow{E}$. Here $\overrightarrow{E}$ is a set of directed edges, where every undirected edge $(a, b) \in E$ is arbitrarily oriented as either the ordered pair (a, b) or (b, a) and included in $\overrightarrow{E}$.

In an electrical flow $i : \overrightarrow{E} \to \mathbb{R}$, for every node other than s or t has zero net external current:

$$\sum_{b \in V\colon (a,b) \in \overrightarrow{E}} i(a, b) - \sum_{b \in V\colon (b,a) \in \overrightarrow{E}} i(b, a) = 0 \qquad \text{for every } a \in V \setminus \{s, t\} .$$

This is known as Kirchhoff's (current) law or flow conservation.

The amount of current $i(a, b)$ along every directed edge (a, b) is determined by the voltages at the endpoints a and b and the resistance of (a, b). The battery assigns to t voltage $v(t) = 0$ and to s voltage $v(s) = v_*$ for some positive constant v_* — the voltage of the battery, for example 1.5 volt. Electrical current flows from a node a of higher voltage to node b of lower voltage, a bit like water current flowing from higher altitude to lower.

If endpoints a and b of a directed edge (a, b) has voltage $v(a)$ and $v(b)$ and resistance $1/w(a, b)$, Ohm's law ($V = IR$) says that

$$v(a) - v(b) = i(a, b)/w(a, b) \qquad \text{for every } (a, b) \in \overrightarrow{E} .$$

The electrical flow is specified by the voltage $v \in \mathbb{R}^V$ and current $i : \overrightarrow{E} \to \mathbb{R}$ satisfying Kirchhoff's law, Ohm's law, and the "boundary conditions" $v(t) = 0$ and $v(s) = v_*$. As we will see, these constraints imply a unique solution to v and i.

2. Laplace equation

Given voltage vector $v \in \mathbb{R}^V$, consider the linear map $B : \mathbb{R}^V \to \mathbb{R}^{\overrightarrow{E}}$ mapping v to the vector $B(v)(a, b) = v(a) - v(b)$ of potential difference across the edge $(a, b) \in \overrightarrow{E}$. In other words, B is an $\overrightarrow{E}$-by-V matrix, whose (a, b)-row is $(\mathbb{1}_a - \mathbb{1}_b)^\top$. Equivalently, the (a, b)-row and c-column of B is

$$B(a, b)(c) = \begin{cases} 1 & \text{if } c = a \\ -1 & \text{if } c = b \\ 0 & \text{otherwise} \end{cases} .$$

Ohm's law implies that given the potential difference $v(a) - v(b)$ across an edge, current $i(a, b) = w(a, b)(v(a) - v(b))$. In other words, given voltage vector $v \in \mathbb{R}^V$, Ohm's law says the corresponding current vector is $i = WBv$, where $W : \mathbb{R}^E \to \mathbb{R}^E$ is the diagonal matrix with weight $w(a, b)$ on the diagonal. It does not matter whether W is a matrix indexed by E or $\overrightarrow{E}$, since the edge weight $w(a, b)$ does not depend on the orientation we have chosen in $\overrightarrow{E}$.

Finally, given current vector $i \in \mathbb{R}^{\overrightarrow{E}}$, its external current at node a is

$$\sum_{b \in V\colon (a,b) \in \overrightarrow{E}} i(a, b) - \sum_{b \in V\colon (b,a) \in \overrightarrow{E}} i(b, a) = (B^\top i)(a) ,$$

because for every directed edge $(a,b) \in \overrightarrow{E}$, its contribution to external current is multiplied by 1 if b is the head, and multiplied by -1 if b is the tail.

In summary, given voltage vector $v \in \mathbb{R}^V$, the external current at every node is the vector $B^\top W B v$. What is the linear transformation $B^\top W B : \mathbb{R}^V \to \mathbb{R}^V$? Expanding,

$$B^\top W B = \sum_{(a,b) \in E} w(a,b)(\mathbb{1}_a - \mathbb{1}_b)(\mathbb{1}_a - \mathbb{1}_b)^\top ,$$

the Laplacian L of the graph H!

The problem of finding voltage vector $v \in \mathbb{R}^V$ satisfying flow conservation means solving the equation $u = Lv$, where the external current vector $u \in \mathbb{R}^V$ satisfies flow conservation: $u(a) = 0$ for every $a \in V \setminus \{s,t\}$. We will also set $u(s) = -u(t)$, so that the net current injected to source node s by the battery equals net current extracted at sink node t. This is an example of a Laplace equation — a system of linear equations whose matrix is a graph Laplacian.

3. Pseudo-inverse

To solve for $u = Lv$ given $u \in \mathbb{R}^V$, one is tempted to multiply L^{-1} on both sides to get $v = L^{-1}u$. This does not work because L has eigenvalue 0 and is not invertible. Fortunately, u is orthogonal to the 0-eigenspace of L.

Since H is connected, eigenvalue 0 of L has multiplicity 1, and the 0-eigenspace is spanned by the eigenvector $\mathbb{1}$. This eigenvector has a physical interpretation in an electrical network: if we shift all voltage by the same constant c to get a new voltage vector $v + c\mathbb{1}$, the net injected current is unchanged. This can be seen from Ohm's law, which says that current along an edge only depends on the potential difference across the endpoints, not their absolute voltage.

Consider the (Moore–Penrose) pseudo-inverse L^+ of L. Any symmetric matrix L and its pseudo-inverse L^+ satisfies

$$LL^+ = L^+L = \Pi ,$$

where Π is the orthogonal projection to the span of L (i.e. subspace orthogonal to $\mathbb{1}$ when L is Laplacian of a connected graph).

Since eigenvalue 0 has multiplicity 1, the Laplacian L is invertible in the subspace orthogonal to $\mathbb{1}$. L^+ is the inverse of L on the subspace orthogonal to $\mathbb{1}$, and L^+ has eigenvalue 0 with eigenvector $\mathbb{1}$. In terms of spectral decomposition using eigenvalues λ_ℓ and eigenvectors ψ_ℓ,

$$L = \sum_\ell \lambda_\ell \psi_\ell \psi_\ell^\top \qquad \Longrightarrow \qquad L^+ = \sum_{\ell :\, \lambda_\ell \neq 0} \frac{1}{\lambda_\ell} \psi_\ell \psi_\ell^\top .$$

Solving the Laplace equation $u = Lv$ amounts to computing $v = L^+u$, given external current $u \in \mathbb{R}^V$. Note that indeed $u \perp \mathbb{1}$, so there is a unique solution vector $v \perp \mathbb{1}$ for every u. We have only enforced $u(s) = -u(t)$ without specifying the magnitude of $u(s)$ or $u(t)$, but there is only one value that can satisfy $v(s) - v(t) = v_*$.

Therefore the constraints of electrical flow yield a unique solution to voltage $v \in \mathbb{R}^V$ and current $i : \overrightarrow{E} \to \mathbb{R}$.

4. Electrical power

A unit s-t flow i is any assignment $i : \overrightarrow{E} \to \mathbb{R}$ of flow value to every directed edge, so that i sends one unit of flow from s to t while satisfying flow conservation constraint, that is,

$$B^\top i = \mathbb{1}_s - \mathbb{1}_t .$$

Given any flow $i : \overrightarrow{E} \to \mathbb{R}$, Joule's law ($P = I^2 R$) says the power loss of i along edge $(a,b) \in E$ is $P(a,b) = i(a,b)^2 / w(a,b)$, and total the power loss of i in H is $\sum_{(a,b)\in E} P(a,b)$.

We can express the total power loss of i as

$$i^\top W^{-1} i .$$

The E-by-E matrix W^{-1} can be interpreted as the diagonal matrix of resistance.

Which unit s-t flow minimizes the total power loss? This is the optimization problem

$$\operatorname{argmin} \left\{ i^\top W^{-1} i \;\middle|\; \text{unit } s\text{-}t \text{ flows } i \right\} .$$

Thompson's principle says the unique minimizer is the electrical flow that satisfies Ohm's law! We now prove Thompson's principle.

Let $f : \overrightarrow{E} \to \mathbb{R}$ be any unit s-t flow, and $i : \overrightarrow{E} \to \mathbb{R}$ the unit s-t flow that satisfies Ohm's law. Consider the circulation $c = f - i$.

Then $B^\top c = B^\top (f - i) = 0$, which means conservation $\sum_{(a,b)\in\overrightarrow{E}} c(a,b) = \sum_{(b,a)\in\overrightarrow{E}} c(b,a)$ at every $a \in V$.

The total power loss of f is

$$f^\top W^{-1} f = (i+c)^\top W^{-1}(i+c) = i^\top W^{-1} i + i^\top W^{-1} c + c^\top W^{-1} i + c^\top W^{-1} c \ .$$

The first term is the total power loss of i, and the last term is positive if $c \neq 0$ ($f \neq i$).

Since i satisfies Ohm's law, it is induced by voltage $v \in \mathbb{R}^V$ so that $i = WBv$, and

$$i^\top W^{-1} c = (WBv)^\top W^{-1} c = v^\top B^\top c = 0 \ .$$

This proves Thompson's principle.

5. Fun facts: Max-Flow and shortest path

If you know the Max-Flow problem (maximizing amount of water current along a pipe network subject to edge capacity constraints), you will recognize that the flow in Max-Flow also satisfies flow conservation. Max-Flow is equivalent to the optimization problem

$$\operatorname{argmin} \left\{ \|W^{-1} i\|_\infty \ \middle|\ \text{unit } s\text{-}t \text{ flows } i \right\} \ ,$$

where the vector $W^{-1} i$ measures, for every edge $(a,b) \in \overrightarrow{E}$, the congestion of the flow i. The Max-Flow value is then $1/\|W^{-1} i_*\|_\infty$ where i_* is the minimizer.

Also, the shortest s-t path problem is equivalent to the optimization problem

$$\operatorname{argmin} \left\{ \|Wi\|_1 \mid \text{unit } s\text{-}t \text{ flows } i \right\} \ .$$

An optimizer i_* can be chosen to be supported on a shortest s-t path.

6. Rayleigh's monotonicity

Consider two resistor networks on the same vertex set V, with resistances $r, r' : \binom{V}{2} \to \mathbb{R}_{\geqslant 0}$. Suppose $r' \geqslant r$ (entrywise for every pair of vertices). Intuitively electrical flow on r' would dissipate more power than on r. This is Rayleigh's monotonicity principle.

More precisely, consider two vertices s and t in V. Let f be the unit electrical flow from s to t when resistances are r. Then total power loss of f under resistance r is

$$f^\top \mathrm{Diag}(r) f \ ,$$

where $\mathrm{Diag}(r)$ is the diagonal matrix with resistance r on the diagonal. Similarly, let f' be the unit electrical flow from s to t when resistances are r'. Then

$$(f')^\top \mathrm{Diag}(r') f' \geqslant (f')^\top \mathrm{Diag}(r) f' \geqslant f^\top \mathrm{Diag}(r) f \ .$$

The first inequality is due to $\mathrm{Diag}(r') \succcurlyeq \mathrm{Diag}(r)$, and the second inequality is Thompson's principle.