Parsing

CSCI 3130 Formal Languages and Automata Theory

Siu On CHAN

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Chinese University of Hong Kong

Context-free versus regular

Write a CFG for the language $(0+1)^*111$

$$\begin{aligned} S &\to U111 \\ U &\to 0U \mid 1U \mid \varepsilon \end{aligned}$$

Can you do so for every regular language?

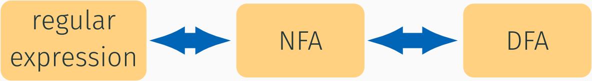
Context-free versus regular

Write a CFG for the language $(0+1)^*111$

$$\begin{aligned} S &\to U111 \\ U &\to 0U \mid 1U \mid \varepsilon \end{aligned}$$

Can you do so for every regular language?

Every regular language is context-free



regular expression ⟷ NFA ⟷ DFA

From regular to context-free

regular expression	$\Rightarrow$ CFG
$\varnothing$	grammar with no rules
ε	$S \rightarrow \varepsilon$
x (alphabet symbol)	$S \rightarrow$ x
$E_1 + E_2$	$S \rightarrow S_1 \mid S_2$
$E_1 E_2$	$S \rightarrow S_1 S_2$
E_1^*	$S \rightarrow S S_1 \mid \varepsilon$

S becomes the new start variable

Context-free versus regular

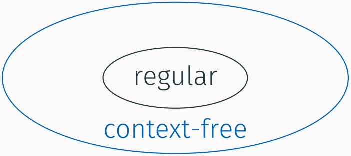
Is every context-free language regular?

Context-free versus regular

Is every context-free language regular?

$S \to 0S1 \mid \varepsilon \qquad L = \{0^n1^n \mid n \geqslant 0\}$

Is context-free but not regular



regular

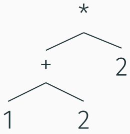
context-free

Ambiguity

Ambiguity

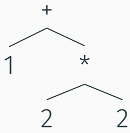
$$E \to E{+}E \mid E{*}E \mid (E) \mid N$$
$$N \to 1 \mid 2$$

1+2*2



✗

$= 6$



$= 5$

A CFG is ambiguous if some string has more than one parse tree

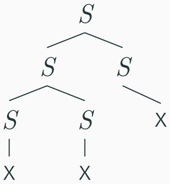
Example

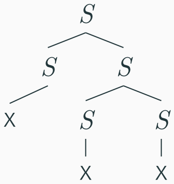
Is $S \to SS \mid$ x ambiguous?

Example

Is $S \to SS \mid \text{x}$ ambiguous?

Yes, because

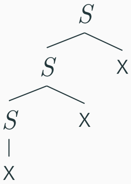




Two parse trees for xxx

Disambiguation

$S \to SS \mid \mathsf{x}$ $\Longrightarrow$ $S \to S\mathsf{x} \mid \mathsf{x}$



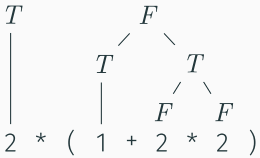
Sometimes we can rewrite the grammar to remove ambiguity

Disambiguation

$$E \to E\texttt{+}E \mid E\texttt{*}E \mid (E) \mid N$$
$$N \to 1 \mid 2$$

+ and * have the same precedence!

Decompose expression into terms and factors



Disambiguation

$$E \rightarrow E\texttt{+}E \mid E\texttt{*}E \mid \texttt{(}E\texttt{)} \mid N$$
$$N \rightarrow \texttt{1} \mid \texttt{2}$$

An expression is a sum of one or more terms

$$E \rightarrow T \mid E\texttt{+}T$$

Each term is a product of one or more factors

$$T \rightarrow F \mid T\texttt{*}F$$

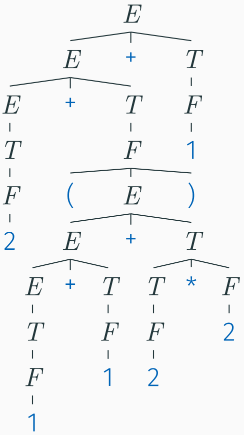
Each factor is a parenthesized expression or a number

$$F \rightarrow \texttt{(}E\texttt{)} \mid \texttt{1} \mid \texttt{2}$$

Parsing example

$$E \rightarrow T \mid E\texttt{+}T$$
$$T \rightarrow F \mid T\texttt{*}F$$
$$F \rightarrow (E) \mid 1 \mid 2$$

Parse tree for
2+(1+1+2*2)+1



Disambiguation

Disambiguation is not always possible because

1. There exists inherently ambiguous languages
   i.e. ambiguous no matter how you rewrite the grammar
2. There is no general procedure for disambiguation

Disambiguation

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1. There exists inherently ambiguous languages
   i.e. ambiguous no matter how you rewrite the grammar
2. There is no general procedure for disambiguation

In programming languages, ambiguity comes from the precedence rules, and we can resolve like in the example

In English, ambiguity is sometimes a problem:

$$\rlap{$\overbrace{\phantom{\text{I look at}}}\overbrace{\phantom{\text{the dog with one eye}}}$}\underbrace{\text{I look at the dog}}\underbrace{\text{with one eye}}$$

Parsing

Parsing

$S \to 0S1 \mid 1S0S \mid T$

$T \to S \mid \varepsilon$

input: 0011

Is $0011 \in L$?

If so, how to build a parse tree with a program?

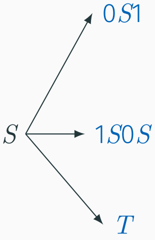
Parsing

$S \to 0S1 \mid 1S0S \mid T$

$T \to S \mid \varepsilon$

input: 0011

Try all derivations?



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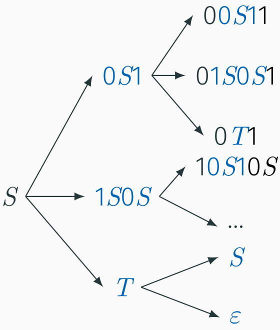


S

$0S1$

$00S11$

$01S0S1$

$0T1$

$1S0S$

$10S10S$

...

T

S

ε

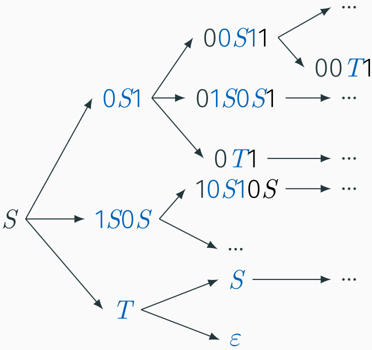
Parsing

$S \rightarrow 0S1 \mid 1S0S \mid T$

$T \rightarrow S \mid \varepsilon$

input: 0011

Try all derivations?



Parsing

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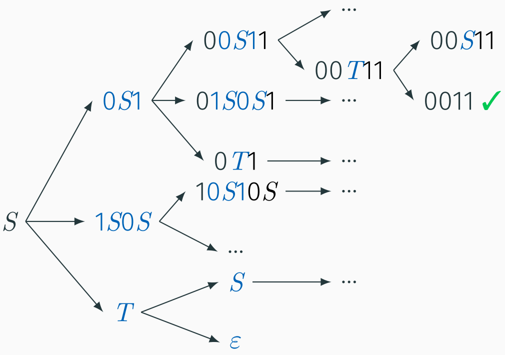
Try all derivations?



This is (part of) the tree of all derivations, not the parse tree

Problems

1. Trying all derivations may take too long
2. If input is not in the language, parsing will never stop

Let’s tackle the 2nd problem

When to stop

$S \rightarrow 0S1 \mid 1S0S \mid T$

$T \rightarrow S \mid \varepsilon$

Idea: Stop when |derived string| > |input|

When to stop

$$S \to 0S1 \mid 1S0S \mid T$$
$$T \to S \mid \varepsilon$$

Idea: Stop when
|derived string| > |input|

Problems:

$$S \Rightarrow 0S1 \Rightarrow 0T1 \Rightarrow 01$$

Derived string may shrink
because of “ε-productions”

When to stop

$$S \to 0S1 \mid 1S0S \mid T$$
$$T \to S \mid \varepsilon$$

Idea: Stop when |derived string| > |input|

Problems:

$S \Rightarrow 0S1 \Rightarrow 0T1 \Rightarrow 01$

Derived string may shrink because of "ε-productions"

$S \Rightarrow T \Rightarrow S \Rightarrow T \Rightarrow \dots$

Derviation may loop because of "unit productions"

Remove ε and unit productions

Note: we will remove all $A \to \varepsilon$ rules, except for start variable A

Removing ε-productions

Goal: remove all $A \to \varepsilon$ rules for every non-start variable A

① If start variable S appears on RHS of a rule

Add a new start variable T
Add the rule $T \to S$

② For every rule $A \to \varepsilon$ where A isn't the (new) start variable

1. Remove the rule $A \to \varepsilon$
2. If you see $B \to \alpha A \beta$
   Add a new rule $B \to \alpha\beta$

$$\begin{aligned}
S &\to ACD \\
A &\to \text{a} \\
B &\to \varepsilon \\
C &\to ED \mid \varepsilon \\
D &\to BC \mid \text{b} \\
E &\to \text{b}
\end{aligned}$$

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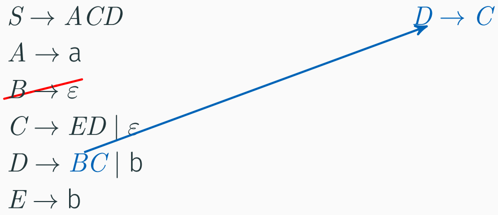

$S \to ACD$
$A \to$ a
~~$B \to \varepsilon$~~
$C \to ED \mid \varepsilon$
$D \to BC \mid$ b
$E \to$ b

$D \to C$

Removing $B \to \varepsilon$

Removing ε-productions

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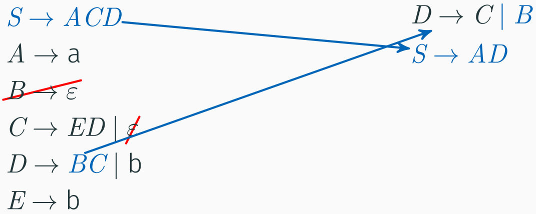

$S \to ACD$

$A \to$ a

~~$B \to \varepsilon$~~

$C \to ED \mid \cancel{\varepsilon}$

$D \to BC \mid$ b

$E \to$ b

$D \to C \mid B$

$S \to AD$

Removing $C \to \varepsilon$

Removing ε-productions

Goal: remove all $A \to \varepsilon$ rules for every non-start variable A

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Add a new start variable T
Add the rule $T \to S$

② For every rule $A \to \varepsilon$ where A isn't the (new) start variable

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$$
\begin{aligned}
S &\to ACD \\
A &\to \mathrm{a} \\
&\cancel{B \to \varepsilon} \\
C &\to ED \mid \cancel{\varepsilon} \\
D &\to BC \mid \mathrm{b} \\
E &\to \mathrm{b}
\end{aligned}
$$

$$
\begin{aligned}
D &\to C \mid B \\
S &\to AD \\
D &\to \varepsilon
\end{aligned}
$$

Removing $C \to \varepsilon$

Removing ε-productions

Goal: remove all $A \to \varepsilon$ rules for every non-start variable A

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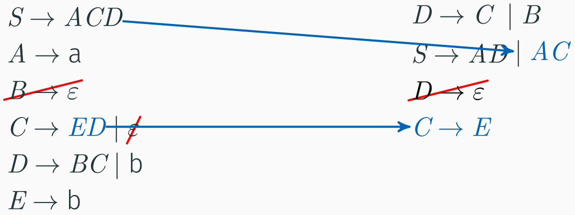

$S \to ACD$
$A \to \text{a}$
$\cancel{B \to \varepsilon}$
$C \to ED \mid \cancel{\varepsilon}$
$D \to BC \mid \text{b}$
$E \to \text{b}$

$D \to C \mid B$
$S \to AD \mid AC$
$\cancel{D \to \varepsilon}$
$C \to E$

Removing $D \to \varepsilon$

Removing ε-productions

Goal: remove all $A \to \varepsilon$ rules for every non-start variable A

① If start variable S appears on RHS of a rule

Add a new start variable T
Add the rule $T \to S$

② For every rule $A \to \varepsilon$ where A isn't the (new) start variable

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$$
\begin{aligned}
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&A \to \text{a} \\
&\cancel{B \to \varepsilon} \\
&C \to ED \mid \cancel{\varepsilon} \\
&D \to BC \mid \text{b} \\
&E \to \text{b}
\end{aligned}
$$

$$
\begin{aligned}
&D \to C \mid B \\
&S \to AD \mid AC \\
&\cancel{D \to \varepsilon} \\
&C \to E \\
&S \to A
\end{aligned}
$$

Removing $D \to \varepsilon$

Eliminating ε-productions

② For every $A \to \varepsilon$ rule where A is not the start variable

1. Remove the rule $A \to \varepsilon$
2. If you see $B \to \alpha A \beta$
   Add a new rule $B \to \alpha\beta$

Do 2. every time A appears

$B \to \alpha A \beta A \gamma$ yields

$B \to \alpha \beta A \gamma \quad B \to \alpha A \beta \gamma$

$B \to \alpha\beta\gamma$

Eliminating ε-productions

② For every $A \to \varepsilon$ rule where A is not the start variable

1. Remove the rule $A \to \varepsilon$
2. If you see $B \to \alpha A \beta$
   Add a new rule $B \to \alpha\beta$

Do 2. every time A appears

$B \to \alpha A \beta A \gamma$ yields

$B \to \alpha\beta A\gamma \qquad B \to \alpha A \beta\gamma$

$B \to \alpha\beta\gamma$

$B \to A$ becomes $B \to \varepsilon$

If $B \to \varepsilon$ was removed earlier, don't add it back

Eliminating unit productions

A unit production is a production of the form

$$A \to B$$

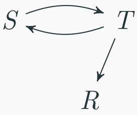
Grammar:

$S \to 0S1 \mid 1S0S \mid T$

$T \to S \mid R \mid \varepsilon$

$R \to 0SR$

Unit production graph:



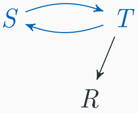
Removing unit productions

① If there is a cycle of unit productions

$$A \to B \to \cdots \to C \to A$$

delete it and replace everything with A
(any variable in the cycle)

$$S \to 0S1 \mid 1S0S \mid T$$
$$T \to S \mid R \mid \varepsilon$$
$$R \to 0SR$$



Removing unit productions

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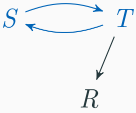
$$A \to B \to \cdots \to C \to A$$

delete it and replace everything with A
(any variable in the cycle)

$S \to 0S1 \mid 1S0S \mid \cancel{T}$

$\cancel{T} \to \cancel{S} \mid R \mid \varepsilon$

$R \to 0SR$



$S \to 0S1 \mid 1S0S$

$S \to R \mid \varepsilon$

$R \to 0SR$

Replace T by S

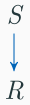
Removal of unit productions

② replace any chain

$$A \to B \to \cdots \to C \to \alpha$$

by $\quad A \to \alpha, \quad B \to \alpha, \quad \cdots, \quad C \to \alpha$

$$\begin{aligned} S &\to 0S1 \mid 1S0S \\ &\quad \mid R \mid \varepsilon \\ R &\to 0SR \end{aligned}$$



S

R

Removal of unit productions

② replace any chain

$$A \to B \to \cdots \to C \to \alpha$$

by $\quad A \to \alpha, \quad B \to \alpha, \quad \cdots, \quad C \to \alpha$

$$\begin{aligned} S &\to 0S1 \mid 1S0S \\ &\mid R \mid \varepsilon \\ R &\to 0SR \end{aligned}$$

$$\begin{tikzcd} S \arrow[d] \\ R \end{tikzcd}$$

$$\begin{aligned} S &\to 0S1 \mid 1S0S \\ &\mid 0SR \mid \varepsilon \\ R &\to 0SR \end{aligned}$$

Replace $\quad S \to R \to 0SR \quad$ by $\quad S \to 0SR, \quad R \to 0SR$

Recap

Problems:

1. Trying all derivations may take too long
2. If input is not in the language, parsing will never stop ✓

Solution to problem 2:

1. Eliminate ε productions
2. Eliminate unit productions

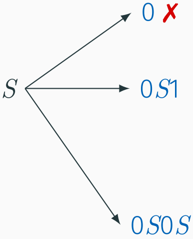
Try all possible derivations but stop parsing when

$|\text{derived string}| > |\text{input}|$

Example

$$S \to 0S1 \mid 0S0S \mid T \qquad\qquad \Longrightarrow \qquad\qquad S \to 0S1 \mid 0S0S \mid 0$$
$$T \to S \mid 0$$

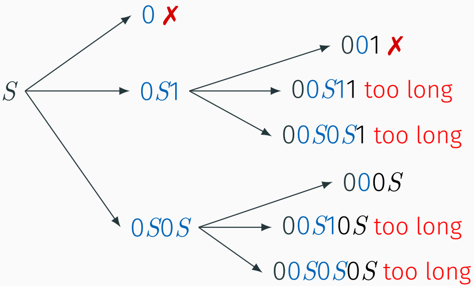
input: 0011



Example

$$S \to 0S1 \mid 0S0S \mid T \qquad\qquad \Longrightarrow \qquad\qquad S \to 0S1 \mid 0S0S \mid 0$$
$$T \to S \mid 0$$

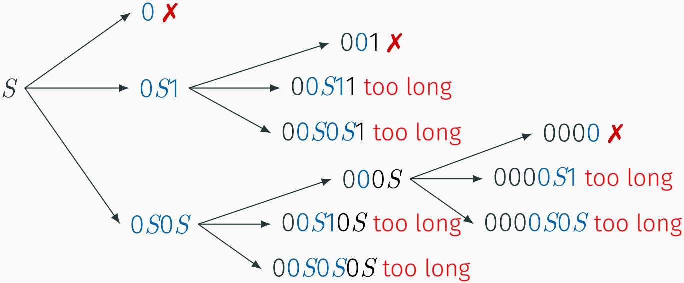
input: 0011



Example

$$\begin{aligned} S &\to 0S1 \mid 0S0S \mid T \\ T &\to S \mid 0 \end{aligned} \quad \Longrightarrow \quad S \to 0S1 \mid 0S0S \mid 0$$

input: 0011



S

0 ✗

$0S1$

001 ✗

$00S11$ too long

$00S0S1$ too long

$0S0S$

$000S$

0000 ✗

$0000S1$ too long

$0000S0S$ too long

$00S10S$ too long

$00S0S0S$ too long

Conclusion: $0011 \notin L$

Problems

1. Trying all derivations may take too long
2. If input is not in the language, parsing will never stop

Preparations

A faster way to parse:

Cocke–Younger–Kasami algorithm

To use it we must perprocess the CFG as follows:

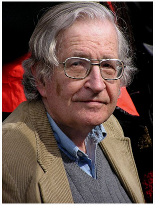
1. Eliminate ε productions
2. Eliminate unit productions
3. Convert CFG to Chomsky Normal Form

Chomsky Normal Form

A CFG is in Chomsky Normal Form if
every production is of one of the following

- $A \to BC$
  (exactly two non-start variables on the right)
- $A \to$ x
  (exactly one terminal on the right)
- $S \to \varepsilon$
  (ε-production only allowed for start variable)

where
A : variable
B and C : non-start variables
x : terminal
S : start variable



Noam Chomsky

Convert to Chomsky Normal Form

$A \to B\text{c}DE$

$\Longrightarrow$ replace terminals with new variables

$A \to BCDE$

$C \to \text{c}$

$\Longrightarrow$ break up sequences with new variables

$A \to BX$

$X \to CY$

$Y \to DE$

$C \to \text{c}$

Cocke–Younger–Kasami algorithm

$$S \rightarrow AB \mid BC$$

$$A \rightarrow BA \mid \text{a}$$

$$B \rightarrow CC \mid \text{b}$$

$$C \rightarrow AB \mid \text{a}$$

Input: $x =$ baaba

let $x[i, \ell] = x_i x_{i+1} \dots x_{i+\ell-1}$

ℓ						
5						
4						
3						
2						
1						
	1	2	3	4	5	i
	b	a	a	b	a	

For every substring $x[i, \ell]$, remember all variables R that derive $x[i, \ell]$

Store in a table $T[i, \ell]$

Cocke–Younger–Kasami algorithm

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$$C \to AB \mid \mathsf{a}$$

Input: $x =$ baaba

let $x[i,\ell] = x_i x_{i+1} \dots x_{i+\ell-1}$

ℓ						
5						
4						
3						
2						
1	B	$A\vert C$	$A\vert C$	B	$A\vert C$	
	1	2	3	4	5	i
	b	a	a	b	a	

For every substring $x[i,\ell]$, remember all variables R that derive $x[i,\ell]$

Store in a table $T[i,\ell]$

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ℓ						
5						
4						
3						
2	$S \mid A$					
1	B	$A \mid C$	$A \mid C$	B	$A \mid C$	
	1	2	3	4	5	i
	b	a	a	b	a	

For every substring $x[i, \ell]$, remember all variables R that derive $x[i, \ell]$

Store in a table $T[i, \ell]$

Cocke–Younger–Kasami algorithm

$$S \to AB \mid BC$$
$$A \to BA \mid \text{a}$$
$$B \to CC \mid \text{b}$$
$$C \to AB \mid \text{a}$$

Input: $x =$ baaba

let $x[i, \ell] = x_i x_{i+1} \dots x_{i+\ell-1}$

ℓ						
5						
4						
3						
2	$S \mid A$	B	$S \mid C$	$S \mid A$		
1	B	$A \mid C$	$A \mid C$	B	$A \mid C$	
	1	2	3	4	5	i
	b	a	a	b	a	

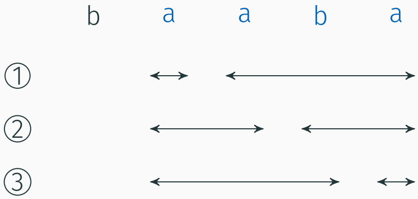
For every substring $x[i, \ell]$, remember all variables R that derive $x[i, \ell]$

Store in a table $T[i, \ell]$

Computing $T[i, \ell]$ for $\ell \geqslant 2$

Example: to compute $T[2, 4]$

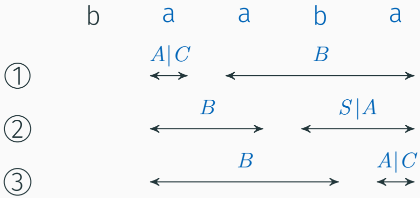
Try all possible ways to split $x[2, 4]$ into two substrings



Computing $T[i,\ell]$ for $\ell \geqslant 2$

Example: to compute $T[2,4]$

Try all possible ways to split $x[2,4]$ into two substrings

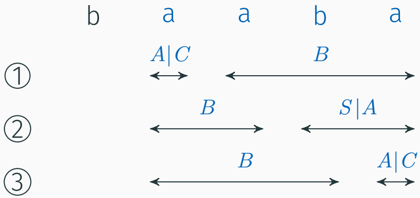


Look up entries regarding shorter substrings previously computed

Computing $T[i,\ell]$ for $\ell \geqslant 2$

Example: to compute $T[2,4]$

Try all possible ways to split $x[2,4]$ into two substrings



Look up entries regarding shorter substrings previously computed

$$\begin{aligned} S &\to AB \mid BC \\ A &\to BA \mid \text{a} \\ B &\to CC \mid \text{b} \\ C &\to AB \mid \text{a} \end{aligned}$$

$$T[2,4] = S|A|C$$

Cocke–Younger–Kasami algorithm

$$S \to AB \mid BC$$
$$A \to BA \mid \text{a}$$
$$B \to CC \mid \text{b}$$
$$C \to AB \mid \text{a}$$

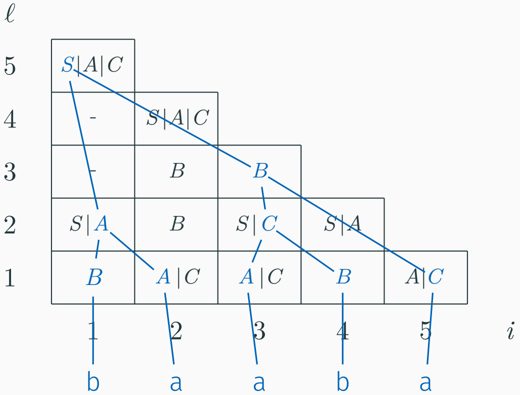
Input: $x =$ baaba

ℓ						
5	$S \mid A \mid C$					
4	-	$S \mid A \mid C$				
3	-	B	B			
2	$S \mid A$	B	$S \mid C$	$S \mid A$		
1	B	$A \mid C$	$A \mid C$	B	$A \mid C$	
	1	2	3	4	5	i
	b	a	a	b	a	

Cocke–Younger–Kasami algorithm

$$S \to AB \mid BC$$
$$A \to BA \mid \text{a}$$
$$B \to CC \mid \text{b}$$
$$C \to AB \mid \text{a}$$

Input: $x = $ baaba



ℓ					
5	$S\|A\|C$				
4	-	$S\|A\|C$			
3	-	B	B		
2	$S\|A$	B	$S\|C$	$S\|A$	
1	B	$A\|C$	$A\|C$	B	$A\|C$
i	1	2	3	4	5
	b	a	a	b	a

Get parse tree by tracing back derivations