Context-free Grammars

CSCI 3130 Formal Languages and Automata Theory

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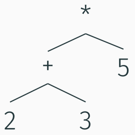
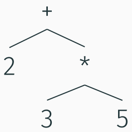
Precedence in Arithmetic Expressions

```
bash$ python
Python 2.7.9 (default, Apr  2 2015, 15:33:21)
>>> 2+3*5
17
```



$= 25$

or



$= 17$

Grammars describe meaning

EXPR → EXPR + TERM

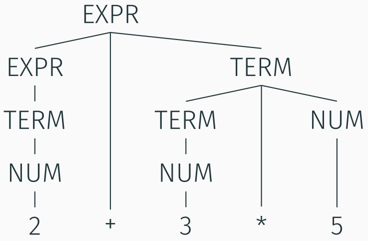
EXPR → TERM

TERM → TERM * NUM

TERM → NUM

NUM → 0-9

rules for valid (simple) arithmetic expressions



Rules always yield the correct meaning

Grammar of English

SENTENCE → NOUN-PHRASE VERB-PHRASE

$$\underbrace{\text{a girl}}_{\text{NOUN-PHRASE}}\ \underbrace{\text{likes the boy}}_{\text{VERB-PHRASE}}$$

NOUN-PHRASE → A-NOUN

or → A-NOUN PREP-PHRASE

$$\underbrace{\text{a girl}}_{\text{A-NOUN}} \qquad \underbrace{\text{a girl}}_{\text{A-NOUN}}\ \underbrace{\text{with a flower}}_{\text{PREP-PHRASE}}$$

Grammar of English

NOUN-PHRASE → A-NOUN

or → A-NOUN PREP-PHRASE

$$\underbrace{\text{a girl}}_{\text{A-NOUN}} \qquad \underbrace{\text{a girl}}_{\text{A-NOUN}}\ \underbrace{\text{with a flower}}_{\text{PREP-PHRASE}}$$

PREP-PHRASE → PREP NOUN-PHRASE

$$\underbrace{\text{with}}_{\text{PREP}}\ \underbrace{\text{a flower}}_{\text{NOUN-PHRASE}}$$

Grammar of English

NOUN-PHRASE $\rightarrow$ A-NOUN

or $\rightarrow$ A-NOUN PREP-PHRASE

a girl (A-NOUN)

a girl (A-NOUN) with a flower (PREP-PHRASE)

PREP-PHRASE $\rightarrow$ PREP NOUN-PHRASE

with (PREP) a flower (NOUN-PHRASE)

Recursive structure

Grammar of (parts of) English

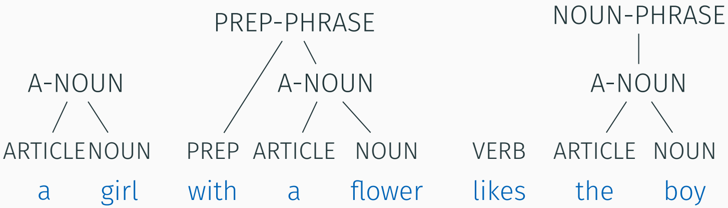
$$
\begin{aligned}
\text{SENTENCE} &\rightarrow \text{NOUN-PHRASE VERB-PHRASE} \\
\text{NOUN-PHRASE} &\rightarrow \text{A-NOUN} \\
\text{NOUN-PHRASE} &\rightarrow \text{A-NOUN PREP-PHRASE} \\
\text{VERB-PHRASE} &\rightarrow \text{CMPLX-VERB} \\
\text{VERB-PHRASE} &\rightarrow \text{CMPLX-VERB PREP-PHRASE} \\
\text{PREP-PHRASE} &\rightarrow \text{PREP A-NOUN} \\
\text{A-NOUN} &\rightarrow \text{ARTICLE NOUN} \\
\text{CMPLX-VERB} &\rightarrow \text{VERB NOUN-PHRASE} \\
\text{CMPLX-VERB} &\rightarrow \text{VERB}
\end{aligned}
$$

$$
\begin{aligned}
\text{ARTICLE} &\rightarrow \text{a} \\
\text{ARTICLE} &\rightarrow \text{the} \\
\text{NOUN} &\rightarrow \text{boy} \\
\text{NOUN} &\rightarrow \text{girl} \\
\text{NOUN} &\rightarrow \text{flower} \\
\text{VERB} &\rightarrow \text{likes} \\
\text{VERB} &\rightarrow \text{touches} \\
\text{VERB} &\rightarrow \text{sees} \\
\text{PREP} &\rightarrow \text{with}
\end{aligned}
$$

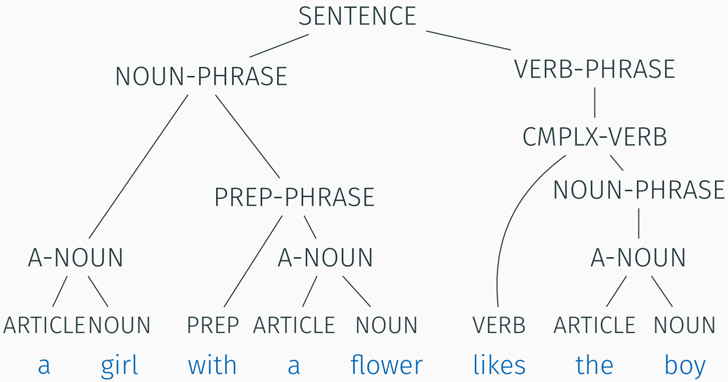
The meaning of sentences

ARTICLE	NOUN	PREP	ARTICLE	NOUN	VERB	ARTICLE	NOUN
a	girl	with	a	flower	likes	the	boy

The meaning of sentences



The meaning of sentences



Context-free grammar

$$A \to 0A1$$
$$A \to B$$
$$B \to \#$$

A, B are variables

0, 1 are terminals

$A \to 0A1$ is a production

A is the start variable

$$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 000A111 \Rightarrow 000B111 \Rightarrow 000\#111$$

derivation

Context-free grammar

A context-free grammar is given by (V, Σ, R, S) where

- V is a finite set of variables or non-terminals
- Σ is a finite set of terminals
- R is a finite set of productions or substitution rules of the form

$$A \to \alpha$$

  A is a variable and α is a string of variables and terminals
- $S \in V$ is a variable called the start variable

Notation and conventions

$E \to E{+}E$

$E \to (E)$

$E \to N$

$N \to 0N$

$N \to 1N$

$N \to 0$

$N \to 1$

Variables: E, N
Terminals: +, (,), 0, 1
Start variable: E

shorthand:

$E \to E{+}E \mid (E) \mid N$

$N \to 0N \mid 1N \mid 0 \mid 1$

conventions:

variables in UPPERCASE
start variable comes first

Derivation

derivation: a sequential application of productions

$$
\begin{aligned}
E &\Rightarrow E\text{+}E \\
&\Rightarrow (E)\text{+}E \\
&\Rightarrow (E)\text{+}N \\
&\Rightarrow (E)\text{+}1 \\
&\Rightarrow (E\text{+}E)\text{+}1 \\
&\Rightarrow (N\text{+}E)\text{+}1 \\
&\Rightarrow (N\text{+}N)\text{+}1 \\
&\Rightarrow (N\text{+}1N)\text{+}1 \\
&\Rightarrow (N\text{+}10)\text{+}1 \\
&\Rightarrow (1\text{+}10)\text{+}1
\end{aligned}
$$

derivation ↓

$$
\begin{aligned}
E &\to E\text{+}E \mid (E) \mid N \\
N &\to 0N \mid 1N \mid 0 \mid 1
\end{aligned}
$$

$\alpha \Rightarrow \beta$
application of one production

Derivation

derivation: a sequential application of productions

$$
\begin{aligned}
E &\Rightarrow E\text{+}E \\
&\Rightarrow (E)\text{+}E \\
&\Rightarrow (E)\text{+}N \\
&\Rightarrow (E)\text{+}1 \\
&\Rightarrow (E\text{+}E)\text{+}1 \\
&\Rightarrow (N\text{+}E)\text{+}1 \\
&\Rightarrow (N\text{+}N)\text{+}1 \\
&\Rightarrow (N\text{+}1N)\text{+}1 \\
&\Rightarrow (N\text{+}10)\text{+}1 \\
&\Rightarrow (1\text{+}10)\text{+}1
\end{aligned}
$$

derivation ↓

$$E \to E\text{+}E \mid (E) \mid N$$
$$N \to 0N \mid 1N \mid 0 \mid 1$$

$\alpha \Rightarrow \beta$
application of one production

$E \overset{*}{\Rightarrow} (1\text{+}10)\text{+}1$

$\alpha \overset{*}{\Rightarrow} \beta$ derivation

Context-free languages

The language of a CFG is the set of all strings at the end of a derivation

$$L(G) = \{w \in \Sigma^* \mid S \overset{*}{\Rightarrow} w\}$$

Questions we will ask:

I give you a CFG, what is the language?

I give you a language, write a CFG for it

Analysis example 1

$$A \to 0A1 \mid B$$
$$B \to \#$$

Can you derive:

00#11

#

00#111

00##11

Analysis example 1

$$A \to 0A1 \mid B$$
$$B \to \#$$

Can you derive:

00#11	$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 00B11 \Rightarrow 00\#11$
#	$A \Rightarrow B \Rightarrow \#$
00#111	
00##11	

Analysis example 1

$$A \to 0A1 \mid B$$
$$B \to \#$$

$$L(G) = \{0^n\#1^n \mid n \geqslant 0\}$$

Can you derive:

00#11	$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 00B11 \Rightarrow 00\#11$
#	$A \Rightarrow B \Rightarrow \#$
00#111	No: uneven number of 0s and 1s
00##11	No: too many #

Analysis example 2

$$S \to SS \mid (S) \mid \varepsilon$$

Can you derive

() (()())

Analysis example 2

$$S \to SS \mid (S) \mid \varepsilon$$

Can you derive

$()$ $\qquad$ $(()())$

$$\begin{aligned} S &\Rightarrow (S) \\ &\Rightarrow () \end{aligned}$$

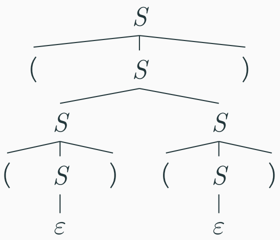
$$\begin{aligned} S &\Rightarrow (S) \\ &\Rightarrow (SS) \\ &\Rightarrow ((S)S) \\ &\Rightarrow ((S)(S)) \\ &\Rightarrow (()(S)) \\ &\Rightarrow (()()) \end{aligned}$$

Parse trees

$$S \to SS \mid (S) \mid \varepsilon$$

A parse tree gives a more compact representation

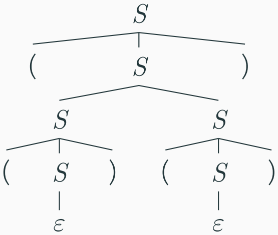
$$\begin{aligned} S &\Rightarrow (S) \\ &\Rightarrow (SS) \\ &\Rightarrow ((S)S) \\ &\Rightarrow ((S)(S)) \\ &\Rightarrow (()(S)) \\ &\Rightarrow (()()) \end{aligned}$$



Parse trees

$$
\begin{aligned}
S &\Rightarrow (S) \\
&\Rightarrow (SS) \\
&\Rightarrow ((S)S) \\
&\Rightarrow ((S)(S)) \\
&\Rightarrow (()(S)) \\
&\Rightarrow (()())
\end{aligned}
$$

$$
\begin{aligned}
S &\Rightarrow (S) \\
&\Rightarrow (SS) \\
&\Rightarrow ((S)S) \\
&\Rightarrow (()S) \\
&\Rightarrow (()(S)) \\
&\Rightarrow (()())
\end{aligned}
$$



$$
\begin{aligned}
S &\Rightarrow (S) \\
&\Rightarrow (SS) \\
&\Rightarrow (S(S)) \\
&\Rightarrow ((S)(S)) \\
&\Rightarrow (()(S)) \\
&\Rightarrow (()())
\end{aligned}
$$

$$
\begin{aligned}
S &\Rightarrow (S) \\
&\Rightarrow (SS) \\
&\Rightarrow (S(S)) \\
&\Rightarrow (S()) \\
&\Rightarrow ((S)()) \\
&\Rightarrow (()())
\end{aligned}
$$

One parse tree can represent many derivations

Analysis example 2

$$S \to SS \mid (S) \mid \varepsilon$$

Can you derive

$(()()$

$())(()$

Analysis example 2

$$S \to SS \mid (S) \mid \varepsilon$$

Can you derive

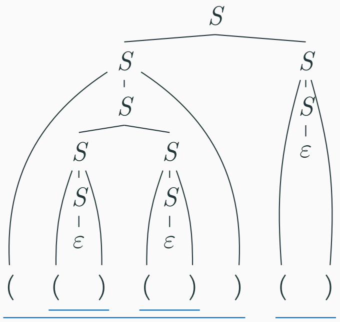
(()() No: uneven number of (and)

())(() No: some prefix has too many)

Analysis example 2

$$S \to SS \mid (S) \mid \varepsilon$$

$L(G) = \{w \mid w$ has the same number of (and)
no prefix of w has more) than (}



Parsing rules:

Divide w into blocks with same number of (and)

Each block is in $L(G)$

Parse each block recursively

Design example 1

$$L = \{0^n 1^n \mid n \geqslant 0\}$$

These strings have a recursive structure

00001111

000111

0011

01

ε

Design example 1

$$L = \{0^n 1^n \mid n \geqslant 0\}$$

These strings have a recursive structure

00001111

000111

0011

01

ε

$$S \to 0S1 \mid \varepsilon$$

Design example 2

$$L = \{0^n1^n0^m1^m \mid n \geqslant 0, m \geqslant 0\}$$

Examples:
010011
00110011
000111

Design example 2

$$L = \{0^n 1^n 0^m 1^m \mid n \geqslant 0, m \geqslant 0\}$$

Examples:
010011
00110011
000111

These strings have two parts:

$$L = L_1 L_2$$
$$L_1 = \{0^n 1^n \mid n \geqslant 0\}$$
$$L_2 = \{0^m 1^m \mid m \geqslant 0\}$$

rules for L_1 : $S_1 \rightarrow 0S_1 1 \mid \varepsilon$
L_2 is the same as L_1

$$S \rightarrow S_1 S_1$$
$$S_1 \rightarrow 0S_1 1 \mid \varepsilon$$

Design example 3

$$L = \{0^n 1^m 0^m 1^n \mid n \geqslant 0, m \geqslant 0\}$$

Examples:
011001
0011
1100
00110011

Design example 3

$$L = \{0^n 1^m 0^m 1^n \mid n \geqslant 0, m \geqslant 0\}$$

Examples:
011001
0011
1100
00110011

These strings have a nested structure:

outer part: $0^n 1^n$

inner part: $1^m 0^m$

$$S \to 0S1 \mid I$$
$$I \to 1I0 \mid \varepsilon$$

Design example 4

$L = \{x \mid x$ has (at least) two nonempty 0-blocks with the same number of 0s$\}$

01011, 001011001, 10010101000
allowed

11001000, 01111
not allowed

Design example 4

$L = \{x \mid x$ has (at least) two nonempty 0-blocks with the same number of 0s$\}$

01011, 001011001, 10010101000 — allowed

11001000, 01111 — not allowed

$$\underbrace{1\,0\,0\,1}_{\substack{\text{initial part}\\ A}}\underbrace{0\,0\,1\,1\,0\,1\,0\,0}_{\substack{\text{middle part}\\ B}}\underbrace{1\,0\,1\,1\,0}_{\substack{\text{final part}\\ C}}$$

A: cannot end in 0

C: cannot begin with 0

Design example 4

$$\underbrace{1\,0\,0\,1}_{A}\,\underbrace{0\,0\,1\,1\,0\,1\,0\,0}_{B}\,\underbrace{1\,0\,1\,1\,0}_{C}$$

$S \to ABC$

$A \to \varepsilon \mid U1$

$U \to 0U \mid 1U \mid \varepsilon$

$C \to \varepsilon \mid 1U$

A: ε, or ends in 1

C: ε, or begins with 1

U: any string

Design example 4

$$\underbrace{1\,0\,0\,1}_{A}\underbrace{0\,0\,1\,1\,0\,1\,0\,0}_{B}\underbrace{1\,0\,1\,1\,0}_{C}$$

$$S \to ABC$$
$$A \to \varepsilon \mid U1$$
$$U \to 0U \mid 1U \mid \varepsilon$$
$$C \to \varepsilon \mid 1U$$
$$B \to 0D0 \mid 0B0$$
$$D \to 1U1 \mid 1$$

A: ε, or ends in 1

C: ε, or begins with 1

U: any string

B has a recursive structure

$$0\,0\,\overbrace{1\,1\,0\,1}^{D}\,0\,0$$

same number of 0s

at least one 0

D: begins and ends in 1