NP-completeness

CSCI 3130 Formal Languages and Automata Theory

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Polynomial-time reductions

When we say

"Independent-Set is at least as hard as Clique"

What does that mean?

We mean

If Clique cannot be decided by a polynomial-time Turing machine, then neither does Independent-Set

If Independent-Set can be decided by a polynomial-time Turing machine, then so does Clique

Similar to the reductions we saw in the past 4-5 lectures, but with the additional restriction of polynomial-time

Polynomial-time reductions

$$\begin{aligned}\textsc{Clique} &= \{\langle G, k\rangle \mid \text{Graph } G \text{ has a clique of } k \text{ vertices}\}\\ \textsc{Independent-Set} &= \{\langle G, k\rangle \mid \text{Graph } G \text{ has having}\\ &\qquad \text{an independent set of } k \text{ vertices}\}\end{aligned}$$

Theorem

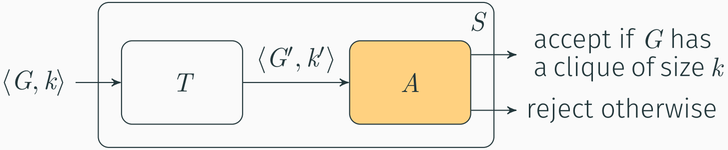
If Independent-Set has a polynomial-time Turing machine, so does Clique

If INDEPENDENT-SET has a polynomial-time Turing machine, so does CLIQUE

Proof

Suppose INDEPENDENT-SET is decided by a poly-time TM A

We want to build a TM S that uses A to solve CLIQUE



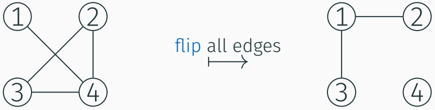
Reducing Clique to Independent-Set

We look for a polynomial-time Turing machine T that turns the question

“Does G have a clique of size k?”

into

“Does G' have an independent set (IS) of size k'?”



Graph G Graph G'

clique of size k $\xleftrightarrow{k=k'}$ IS of size k'

Reducing Clique to Independent-Set

On input $\langle G, k \rangle$

Construct G' by flipping all edges of G

Set $k' = k$

Output $\langle G', k' \rangle$

$$\langle G, k \rangle \longrightarrow \boxed{T} \longrightarrow \langle G', k' \rangle$$

Cliques in G $\quad\longleftrightarrow\quad$ Independent sets in G'

- If G has a clique of size k
  then G' has an independent set of size k
- If G does not have a clique of size k
  then G' does not have an independent set of size k

Reduction recap

We showed that

If Independent-Set is decidable by a polynomial-time Turing machine, so is Clique

by converting any Turing machine for Independent-Set into one for Clique

To do this, we came up with a reduction that transforms instances of Clique into ones of Independent-Set

Polynomial-time reductions

Language L polynomial-time reduces to L' if

there exists a polynomial-time Turing machine T that takes an instance x of L into an instance y of L' such that

$$x \in L \quad \text{if and only if} \quad y \in L'$$

Clique		IS
L		L'
	$\longrightarrow \boxed{T} \longrightarrow$	
$x = \langle G, k \rangle$		$y = \langle G', k' \rangle$
$x \in L$		$y \in L'$
G has a clique of size k		G' has an IS of size k

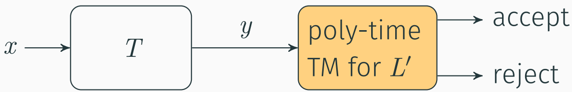
The meaning of reductions

L reduces to L' means L is no harder than L'

If we can solve L', then we can also solve L

Therefore

If L polynomial-time reduces to L' and $L' \in$ P, then $L \in$ P



Direction of reduction

Pay attention to the direction of reduction

"A is no harder than B" and "B is no harder than A"

have completely different meanings

It is possible that L reduces to L' and L' reduces to L

That means L and L' are as hard as each other

For example, IS and CLIQUE reduce to each other

Boolean formula satisfiability

A boolean formula is an expression made up of variables, ANDs, ORs, and negations, like

$$\phi = (x_1 \vee \overline{x}_2) \wedge (x_2 \vee \overline{x}_3 \vee x_4) \wedge (\overline{x}_1)$$

Task: Assign TRUE/FALSE values to variables so that the formula evaluates to true

e.g. $x_1 \mapsto \mathsf{F} \qquad x_2 \mapsto \mathsf{F} \qquad x_3 \mapsto \mathsf{T} \qquad x_4 \mapsto \mathsf{T}$

Given a formula, decide whether such an assignment exist

3SAT

$$\text{SAT} = \{\langle\phi\rangle \mid \phi \text{ is a satisfiable Boolean formula}\}$$

$$\text{3SAT} = \{\langle\phi\rangle \mid \phi \text{ is a satisfiable Boolean formula conjunctive normal form with} \leqslant 3 \text{ literals per clause}\}$$

literal: x_i or $\overline{x}_i$

Conjuctive Normal Form (CNF): AND of ORs of literals

3CNF: CNF with $\leqslant 3$ literals per clause (repetitions allowed)

$$(\underbrace{\overline{x}_1}_{\text{literal}} \vee x_2 \vee \overline{x}_2) \wedge \underbrace{(\overline{x}_2 \vee x_3 \vee x_4)}_{\text{clause}}$$

3SAT is in NP

$$\phi = (x_1 \vee \overline{x}_2) \wedge (x_2 \vee \overline{x}_3 \vee x_4) \wedge (\overline{x}_1)$$

Finding a solution:

Try all possible assignments

FFFF	FTFF	TFFF	TTFF
FFFT	FTFT	TFFT	TTFT
FFTF	FTTF	TFTF	TTTF
FFTT	FTTT	TFTT	TTTT

For n variables, there are 2^n possible assignments

Takes exponential time

Verifying a solution:

substitute

$x_1 \mapsto$ F $\quad x_2 \mapsto$ F

$x_3 \mapsto$ T $\quad x_4 \mapsto$ T

evaluating the formula

$\phi = (\text{F} \vee \text{T}) \wedge (\text{F} \vee \text{F} \vee \text{T}) \wedge (\text{T})$

can be done in linear time

Cook–Levin theorem

Every $L \in$ NP polynomial-time reduces to SAT

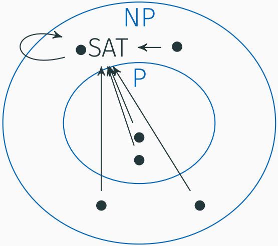
SAT $= \{\langle\phi\rangle \mid \phi$ is a satisfiable Boolean formula$\}$

e.g. $\phi = (x_1 \vee \overline{x}_2) \wedge (x_2 \vee \overline{x}_3 \vee x_4) \wedge (\overline{x}_1)$

Every problem in NP is no harder than SAT

But SAT itself is in NP, so SAT must be the “hardest problem” in NP

If SAT $\in$ P, then P $=$ NP



NP-completeness

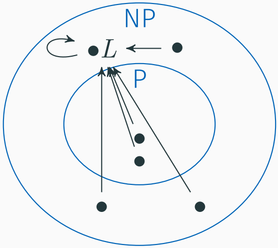
A language L is NP-hard if:

For every N in NP, N polynomial-time reduces to L

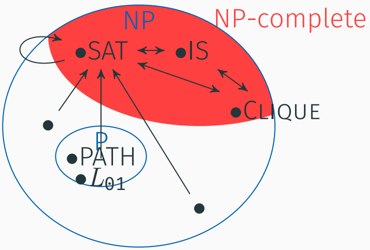
A language L is NP-complete if L is in NP and L is NP-hard

Cook–Levin theorem

SAT is NP-complete



Our (conjectured) picture of NP



$A \to B$: A polynomial-time reduces to B

In practice, most NP problems are either in P (easy) or NP-complete (probably hard)

Interpretation of Cook–Levin theorem

Optimistic:

If we manage to solve SAT, then we can also solve CLIQUE and many other

Pessimistic:

Since we believe $P \neq NP$, it is unlikely that we will ever have a fast algorithm for SAT

Ubiquity of NP-complete problems

We saw a few examples of NP-complete problems, but there are many more

Surprisingly, most computational problems are either in P or NP-complete

By now thousands of problems have been identified as NP-complete

Reducing IS to VC

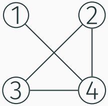
$$\langle G, k\rangle \longrightarrow \boxed{T} \longrightarrow \langle G', k'\rangle$$

G has an IS of size k $\quad\longleftrightarrow\quad$ G' has a VC of size k'

Example

Independent sets:

$\emptyset, \{1\}, \{2\}, \{3\}, \{4\},$
$\{1, 2\}, \{1, 3\}$



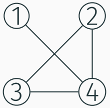
vertex covers:

$\{2, 4\}, \{3, 4\},$
$\{1, 2, 3\}, \{1, 2, 4\},$
$\{1, 3, 4\}, \{2, 3, 4\},$
$\{1, 2, 3, 4\}$

Reducing IS to VC

Claim

S is an independent set if and only if $\overline{S}$ is a vertex cover



Proof:

S is an independent set

$\Updownarrow$

no edge has both endpoints in S

$\Updownarrow$

every edge has an endpoint in $\overline{S}$

$\Updownarrow$

$\overline{S}$ is a vertex cover

IS	VC
$\emptyset$	$\{1,2,3,4\}$
$\{1\}$	$\{2,3,4\}$
$\{2\}$	$\{1,3,4\}$
$\{3\}$	$\{1,2,4\}$
$\{4\}$	$\{1,2,3\}$
$\{1,2\}$	$\{3,4\}$
$\{1,3\}$	$\{2,4\}$

Reducing IS to VC

$$\langle G, k \rangle \longrightarrow \boxed{T} \longrightarrow \langle G', k' \rangle$$

T: On input $\langle G, k \rangle$

Output $\langle G, n-k \rangle$

G has an IS of size $k \quad \longleftrightarrow \quad G$ has a VC of size $n-k$

Overall sequence of reductions:

SAT $\to$ 3SAT $\to$ Clique $\xrightarrow{✓}$ IS $\xrightarrow{✓}$ VC

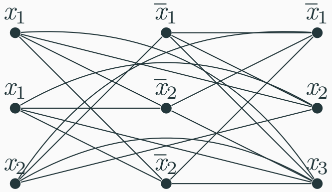
$$\text{3SAT} = \{\phi \mid \phi \text{ is a satisfiable Boolean formula in 3CNF}\}$$
$$\textsc{Clique} = \{\langle G, k\rangle \mid G \text{ is a graph having a clique of } k \text{ vertices}\}$$

$$\text{3CNF formula } \phi \longrightarrow \boxed{T} \longrightarrow \langle G, k\rangle$$

$$\phi \text{ is satisfiable} \quad \longleftrightarrow \quad G \text{ has a clique of size } k$$

Example:

$$\phi = (x_1 \lor x_1 \lor x_2) \land (\overline{x}_1 \lor \overline{x}_2 \lor \overline{x}_2) \land (\overline{x}_1 \lor x_2 \lor x_3)$$



One vertex for each literal occurrence

One edge for each consistent pair across different groups

(not opposite literals of the same variable)

Reducing 3SAT to Clique

$$\text{3CNF formula } \phi \longrightarrow \boxed{T} \longrightarrow \langle G, k \rangle$$

T: On input ϕ, where ϕ is a 3CNF formula with m clauses

Construct the following graph G:

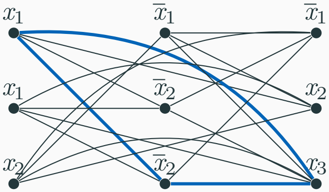
- G has $3m$ vertices, divided into m groups
  - One for each literal occurrence in ϕ
- If vertices u and v are in different groups and consistent
  - Add an edge (u, v)

Output $\langle G, m \rangle$

Reducing 3SAT to CLIQUE

3CNF formula $\phi \longrightarrow \boxed{T} \longrightarrow \langle G, k \rangle$

ϕ is satisfiable $\quad \longleftrightarrow \quad$ G has a clique of size m



$$\phi = (\underset{\mathsf{T}}{x_1} \vee \underset{\mathsf{T}}{x_1} \vee \underset{\mathsf{F}}{x_2}) \wedge (\underset{\mathsf{F}}{\overline{x}_1} \vee \underset{\mathsf{T}}{\overline{x}_2} \vee \underset{\mathsf{T}}{\overline{x}_2}) \wedge (\underset{\mathsf{F}}{\overline{x}_1} \vee \underset{\mathsf{F}}{x_2} \vee \underset{\mathsf{T}}{x_3})$$

Reducing 3SAT to CLIQUE: Summary

3CNF formula $\phi \longrightarrow \boxed{T} \longrightarrow \langle G, k \rangle$

Every satisfying assignment of ϕ gives a clique of size m in G

Conversely, every clique of size m in G gives a satisfying assignment of ϕ

Overall sequence of reductions:

SAT $\to$ 3SAT $\xrightarrow{\checkmark}$ CLIQUE $\xrightarrow{\checkmark}$ IS $\xrightarrow{\checkmark}$ VC

SAT and 3SAT

$$\text{SAT} = \{\phi \mid \phi \text{ is a satisfiable Boolean formula}\}$$

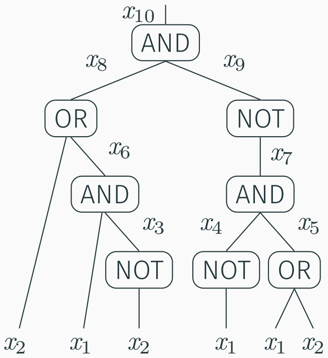
e.g. $((x_1 \vee x_2) \wedge \overline{(x_1 \vee x_2)}) \vee \overline{((x_1 \vee (x_2 \wedge x_3)) \wedge \overline{x}_3)}$

$$\text{3SAT} = \{\phi' \mid \phi' \text{ is a satisfiable 3CNF formula}\}$$

e.g. $(x_1 \vee x_2 \vee x_2) \wedge (x_2 \vee x_3 \vee \overline{x}_4) \wedge (x_2 \vee \overline{x}_3 \vee \overline{x}_5)$

Reducing SAT to 3SAT

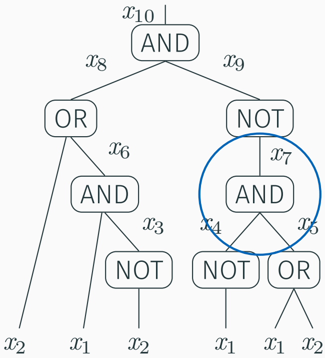
Example: $\phi = (x_2 \vee (x_1 \wedge \bar{x}_2)) \wedge \overline{(\bar{x}_1 \wedge (x_1 \vee x_2))}$



Tree representation of ϕ
Add extra variable to ϕ' for each wire in the tree

Reducing SAT to 3SAT

Example: $\phi = (x_2 \vee (x_1 \wedge \overline{x}_2)) \wedge \overline{(\overline{x}_1 \wedge (x_1 \vee x_2))}$



Tree representation of ϕ
Add extra variable to ϕ' for each wire in the tree

Add clauses to ϕ' for each gate

x_4	x_5	x_7	$x_7 = x_4 \wedge x_5$
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	T
F	F	T	F
F	F	F	T

Clauses added:

$(\overline{x}_4 \vee \overline{x}_5 \vee x_7) \wedge (\overline{x}_4 \vee x_5 \vee \overline{x}_7)$
$(x_4 \vee \overline{x}_5 \vee \overline{x}_7) \wedge (x_4 \vee x_5 \vee \overline{x}_7)$

Reducing SAT to 3SAT

Boolean formula $\phi \longrightarrow \boxed{T} \longrightarrow$ 3CNF formula ϕ'

T: On input $\langle \phi \rangle$, where ϕ is a Boolean formula

Construct and output the following 3CNF formula ϕ'

ϕ' has extra variable $x_{n+1}, \ldots, x_{n+t}$

  one for each gate G_j in ϕ

For each gate G_j, construct the forumla ϕ_j

  forcing the output of G_j to be correct given its inputs

Set $\phi' = \phi_{n+1} \wedge \cdots \wedge \phi_{n+t} \wedge \underbrace{(x_{n+t} \vee x_{n+t} \vee x_{n+t})}_{\text{requires output of } \phi \text{ to be TRUE}}$

Reducing SAT to 3SAT

$$\text{Boolean formula } \phi \longrightarrow \boxed{T} \longrightarrow \text{3CNF formula } \phi'$$

ϕ satisfiable $\longleftrightarrow$ ϕ' satisfiable

Every satisfying assignment of ϕ extends uniquely to a satisfying assignment of ϕ'

In the other direction, in every satisfying assignment of ϕ', the $x_1, \ldots, x_n$ part satisfies ϕ