Notes 3: Duality

This set of notes is based on [3, 2].

1. Weak Duality for SDP

Given an SDP,

$$\begin{array}{rc} \text{maximize} & C \circ X \\ \text{subject to} & A_1 \circ X \le b_1 \\ & \cdots \\ & A_m \circ X \le b_m \\ & X \succeq 0. \end{array}$$

The dual is given by

$$\begin{array}{rc} \text{minimize} & y^T b \\ \text{subject to} & \sum_i y_i A_i \succeq C \\ & y \ge 0, \end{array}$$

where for a vector y, by $y \ge 0$ we mean y is entry-wise nonnegative, and for two matrices A and B, by $A \succeq B$ we mean $A - B \succeq 0$. We now prove weak duality for SDP.

Theorem 1.1 (Weak duality for SDP)**.** *For any primal feasible X and any dual feasible y,*

$$C \circ X \le y^T b.$$

Proof. It suffices to show

$$C \circ X \le \left(\sum_i y_i A_i\right) \circ X = \sum_i y_i (A_i \circ X) \le \sum_i y_i b_i = y^T b.$$

The second equality follows by distributivity of Hadamard inner product over addition. The second inequality follows because X is primal feasible. The second equality follows by expanding the inner product. Thus it remains to show the first inequality. To do that, we first need a lemma.

Lemma 1.2. *$M \succeq 0$ if and only if for any $X \succeq 0$, $M \circ X \ge 0$.*

Proof. (Sufficiency) For any vector v,

$$v^T M v = \sum_i v_i v_j M_{ij} = M \circ (vv^T) \ge 0.$$

Thus $M \succeq 0$.

(Necessity) For any $X \succeq 0$, we write down its eigenvalue decomposition $X = \sum_i \lambda_i v_i v_i^T$ with $\lambda_i \ge 0$ for all i's. Thus

$$M \circ X = \sum_i \lambda_i M \circ v_i v_i^T = \sum_i \lambda_i v_i^T M v_i \ge 0.$$

□

Take $M = \sum_i y_i A_i - C$. By the dual constraint, $M \succeq 0$. The first inequality follows directly by applying Lemma 1.2. □

2. Strong Duality for LP

We will prove separation theorem for closed convex sets and see how it's used to prove strong duality for LP. First, let's recall some definitions.

Definition 2.1 (Convex sets)**.** A set S is said to be convex if for any two points x and y in S and any $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda) y$ is also in S.

Definition 2.2 (Limit points)**.** Given a set S. A point $x \in S$ is a limit point if for any $\delta > 0$,

$$B(x, \delta) \cap (S \backslash \{x\}) \ne \emptyset,$$

where $B(x, \delta)$ is the δ-neighborhood of x, i.e. the open ball of radius δ centered at x.

Definition 2.3 (Closed sets)**.** A set S is said to be closed if it contains all its limit points.

Example 1. As shown in Figure 1, an open disc $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ is not closed. For example, take a point sequence $\{(0, 1 - 1/n)\}_{n=1}^{\infty} \subseteq \mathcal{D}$. $\lim_{n\to\infty}(0, 1 - 1/n) = (0, 1) \notin \mathcal{D}$. By contrast, a closed disc $\bar{\mathcal{D}} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ is closed.

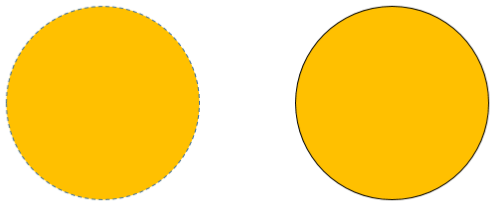

FIGURE 1. Examples of closed and non-closed sets.

Theorem 2.4 (Separation theorem for closed convex sets)**.** *Let S be a closed convex set. Then for any $b \notin S$, there exists y such that, for any $x \in S$, $\langle y, b\rangle > \langle y, x\rangle$.*

Remark 2.5. The separation theorem basically says that if a point doesn't lie in a closed convex set, then there must exist a hyperplane separating it from the set. (See Figure 2.)

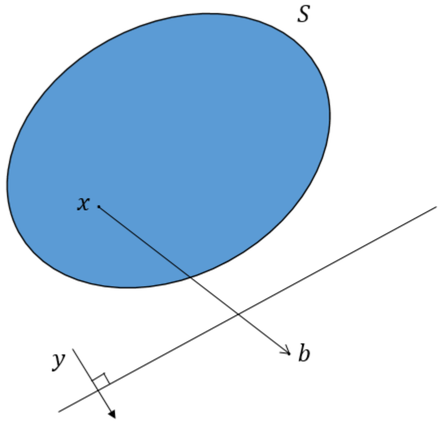

FIGURE 2. Separation theorem.

To prove the above theorem, we need a lemma.

Lemma 2.6 (The best approximation theorem)**.** *Let S be a closed convex set and let $b \notin S$ be a point. Then there exists a unique point $x^* \in S$ which is closest to b, i.e. $\|x^* - b\| = \inf_{x\in S} \|x - b\|$. x^* is called the projection of b onto S or the best approximation of b in S.*

Proof. (Existence. See Figure 3.) Let x_0 be any point in S. Then $S' = B(b, \|x_0 - b\|) \cap S$ is a bounded convex set. As $\|x - b\|$ is a continuous function in x, it can achieve minimum value on S'. Thus there exists $x^* \in S$ such that $\|x^* - b\| = \inf_{x\in S} \|x - b\|$.

(Uniqueness) Suppose for contradiction that there exist two distinct points $x^* \neq x' \in S$ such that $\|x^* - b\| = \|x' - b\| = \inf_{x\in S} \|x - b\|$. Let $y = (x^* + x')/2$. By convexity, $y \in S$. It's easy to verify that $x^* - x' \perp b - y$. Thus by Pythagorean theorem,

$$\|y - b\|^2 = \|x^* - b\|^2 - \|x^* - y\|^2 > \|x^* - b\|^2.$$

The last inequality follows by $\|x^* - y\| > 0$ due to distinction of x^* and x'. This contradicts the definition of x^*. $\square$

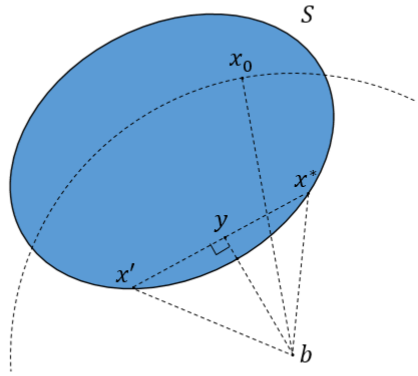

FIGURE 3. The best approximation theorem.

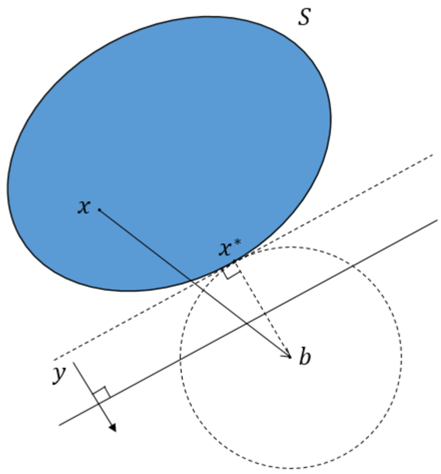

FIGURE 4. Proof of separation theorem from the best approximation theorem.

Proof. (of Theorem 2.4 from Lemma 2.6) Take $y = b - x^*$, where x^* is the projection of b onto S. (See Figure 4.)

It suffices to show that for any $x \in S$

$$\langle y, b - x\rangle = \langle b - x^*, b - x\rangle = \langle b - x^*, b - x^* - (x - x^*)\rangle = \|x^* - b\|^2 - \langle b - x^*, x - x^*\rangle > 0.$$

As $b \notin S$ and S is closed, $\|x^* - b\| > 0$. It remains to show that for any $x \in S$, $\langle b - x^*, x - x^*\rangle \leq 0$. If this is not the case, suppose there exists an $x \in S$ with $\langle b - x^*, x - x^*\rangle > 0$. Let $z = (1-\epsilon)x^* + \epsilon x \in S$, where $\epsilon > 0$ is a small enough constant (see Figure 5).

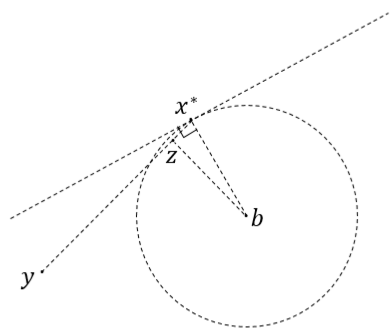

FIGURE 5. Proof of separation theorem.

Then

$$\begin{aligned}\|z-b\|^2 =&\langle (1-\epsilon)x^* + \epsilon x - b, (1-\epsilon)x^* + \epsilon x - b\rangle \\ =&\langle (x^*-b) + \epsilon(x-x^*), (x^*-b) + \epsilon(x-x^*)\rangle \\ =&\|x^*-b\|^2 - 2\epsilon\langle b-x^*, x-x^*\rangle + \epsilon^2\|x^*-x\|^2.\end{aligned}$$

By the choice of x and ϵ,

$$-2\epsilon\langle b-x^*, x-x^*\rangle + \epsilon^2\|x^*-x\|^2 < 0.$$

Thus $\|z-b\| < \|x^*-b\|^2$, which contradicts the definition of x^*. This completes the proof. □

Theorem 2.7 (Farkas' Lemma). *$Ax = b, x \geq 0$ has no solution if and only if there exists y such that $y^T A \geq 0$ and $y^T b < 0$.*

Remark 2.8. Farkas' lemma basically says that a linear system $Ax = b$ has no nonnegative solution if and only if b is not in the cone spanned by column vectors of A, if and only if there exists a hyperplane separating b from the cone. (See Figure 6.)

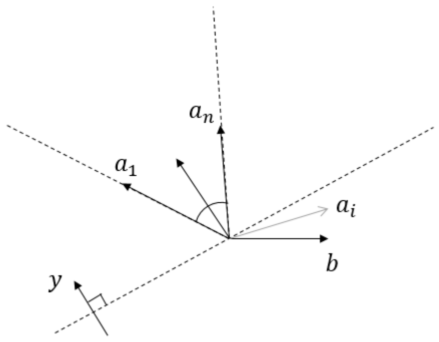

FIGURE 6. Farkas' lemma.

Proof. Suppose A has column vectors $a_1, \cdots, a_n$.

(Sufficiency) If there exists an $x \geq 0$ which is a solution to $Ax = b$, then

$$0 \leq y^T Ax = y^T b < 0,$$

which is a contradiction.

(Necessity) Let $S = \{Ax : x \geq 0\}$. It is easy to verify that S is convex. And we claim (without proof) it's also closed. Take $b \notin S$. By separation theorem, there exists y such that for any $z \in S$, $y^T b < y^T z$, i.e. for any $x \geqslant 0$, $y^T b < y^T Ax$. As $0 \in S$, $y^T b < 0$.

We also claim $y^T A \geq 0$. Otherwise, there exists $i \in [n]$ such that $y^T a_i < 0$. Take x to be a zero vector except that the ith entry is a large enough positive constant. Then $y^T Ax = y^T a_i x_i < y^T b < 0$, which contradicts separation theorem. This completes the proof. □

We did not prove why S is closed. For a proof, see [1, Section 6.5].

3. SDP Duality

To be added later

References

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[2] Bernd Gärtner and Jiri Matousek. Duality and cone programming. In *Approximation Algorithms and Semidefinite Programming*, chapter 4. Springer, 2012.
[3] Lap Chi Lau. Lecture 2 "SDP duality", semidefinite programming and approximation algorithms. 2014.