# Some Exercises on the “Three Basic Techniques”

Hao WU

Department of Computer Science and Engineering
Chinese University of Hong Kong

You have learned three basic techniques in algorithm design:

- Recursion
- Repeating (till success)
- Geometric Series.

In this tutorial, we will discuss some exercises that can be solved using these techniques.

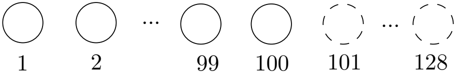
Principle of Recursion

When dealing with a subproblem (same problem but with a smaller input), consider it solved, and use the subproblem's output to continue the algorithm design.

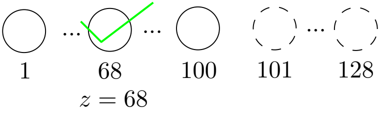
## Exercise 1

Recall that our RAM model has an atomic operation RANDOM$(x, y)$ which, given integers $x, y$, returns an integer chosen uniformly at random from $[x, y]$.

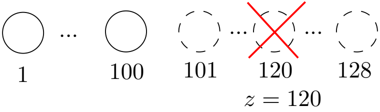
Suppose that you are allowed to call the operation only with $x = 1$ and $y = 128$. Describe an algorithm to obtain a uniformly random number between 1 and 100. Your algorithm must finish in $O(1)$ expected time.




Call RANDOM(1,128) and let $z$ be its return value. Output $z$ if it is in $[1, 100]$.




Otherwise, repeat from the beginning.



We need to call the operator at most twice in expectation because each time $z$ has probability $100/128$ to fall in the range we want. Therefore, our algorithm finishes in $O(1)$ expected time.

## Exercise 2

Suppose that we enforce a harder constraint that you are allowed to call RANDOM$(x, y)$ only with $x = 0$ and $y = 1$. Describe an algorithm to generate a uniformly random number in $[1, n]$ for an arbitrary integer $n$. Your algorithm must finish in $O(\log n)$ expected time.

Suppose $n$ is a power of 2; then how can we use recursion to solve this problem?

1. Set $z =$RANDOM$(x, y)$.
2. If $z = 0$, we have a subproblem: generate a uniformly random number in the first half of the range;
   If $z = 1$, we have a subproblem: generate a uniformly random number in the second half of the range.

Considering the subproblem solved, we finish the algorithm.

## Analysis of the Algorithm

$$
\begin{aligned}
f(1) &= O(1) \\
f(n) &\leq f(n/2) + O(1) \quad \text{, for } n > 1
\end{aligned}
$$

Thus, we have

$$f(n) = O(\log n)$$

**Think:** Why does the algorithm require $n$ to be a power of 2?

Next, we will extend our algorithm to support values of $n$ that are not powers of 2.

First, obtain the smallest power of 2 that is at least $n$.

- Try $1, 2, 4, \ldots$, until reaching $m$ such that $n \leq m < 2n$. This takes $O(\log n)$ time.

We have known how to generate a uniformly random number $y$ in $[1, m]$ in $O(\log n)$ time.

If $y \leq n$, return $y$; otherwise, repeat the algorithm. At most 2 repeats are needed in expectation. The overall time is there $O(\log n)$ in expectation.

**Exercise 3**

Recall the $k$-selection problem:

> You are given a set $S$ of $n$ integers in an array and an integer $k \in [1, n]$. Find the $k$-th smallest integer of $S$.

Suppose there is a deterministic algorithm $\mathcal{A}_1$ which returns the median of $n$ integers in $O(n)$ time. Can you use $\mathcal{A}_1$ as a blackbox to solve $k$-selection in $O(n)$ time?

Consider the following algorithm.

1. Get the median $v$ of $S$ from $\mathcal{A}_1(S)$.
2. Divide $S$ into $S_1$ and $S_2$ where
   - $S_1 =$ the set of elements in $S$ less than or equal to $v$;
   - $S_2 =$ the set of elements in $S$ greater than $v$.
3. If $|S_1| \geq k$, then return $S' = S_1$ and $k' = k$; else return $S' = S_2$ and $k' = k - |S_1|$

Since $\mathcal{A}_1$ is deterministic, we always succeed in obtaining a subproblem with size no larger than $\lceil \frac{|S|}{2} \rceil$.

## Analysis of the Algorithm

$$f(1) = O(1)$$
$$f(n) \leq f(n/2) + O(n)$$

Thus, $f(n) = O(n)$.

What if $\mathcal{A}_1$ returns the $\lceil \frac{4}{5} n \rceil$-th smallest integer of $n$ integers in $O(n)$ time. Can you still use $\mathcal{A}_1$ as a blackbox to solve $k$-selection in $O(n)$ time?

Instead of shrinking the size of subproblem by half, we shrink it by $\frac{4}{5}$.

We can still use $\mathcal{A}_1$ to shrink the problem size by a constant factor. From the geometric series we know that the total cost will be $O(n)$.

**Think:** If $\mathcal{A}_1$ returns the $\lceil \frac{99}{100} n \rceil$-th smallest integer of $n$ integers in $O(n)$ time, can you still use $\mathcal{A}_1$ as a blackbox to solve $k$-selection in $O(n)$ time?

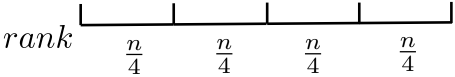
## Exercise 4

Let's still focus on the $k$-selection problem. In the lecture, we shrink the input size of the subproblem into at most $\frac{2}{3}n$. Now, we want to shrink the input size into at most $\frac{n}{2}$. Give an algorithm to achieve the purpose in $O(n)$ expected time.

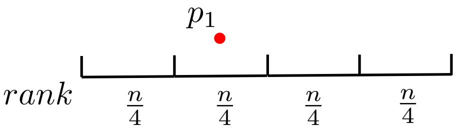
A simple solution: run our "$\frac{2n}{3}$-algorithm" twice. The number of remaining elements becomes at most $\frac{4n}{9}$.

Next, let us look at another way to achieve the purpose, assuming for simplicity that $n$ is a multiple of 4.

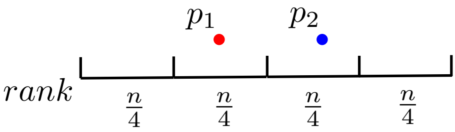
First, divide the rank space into 4 equal partitions.



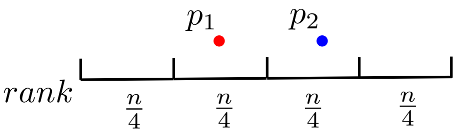

Second, take an element $p_1$ from $S$ uniformly at random. Repeat until $rank(p_1)$ is in range $[\frac{n}{4}, \frac{n}{2}]$.



Third, take an element $p_2$ from $S$ uniformly at random. Repeat until $rank(p_2)$ is in range $[\frac{1}{2}n, \frac{3}{4}n]$.




- If $k \leq rank(p_1)$, set $S' =$ the set of elements in $S$ less than or equal to $p_1$, $k' = k$.
- If $rank(p_1) < k < rank(p_2)$, set $S' =$ the set of elements in $S$ larger than $p_1$ and smaller than $p_2$, $k' = k - rank(p_1)$.
- If $k \geq rank(p_2)$, set $S' =$ the set of elements in $S$ larger than or equal to $p_2$, $k' = k - rank(p_2)$.




In any case, we have $|S'| \leq \frac{n}{4} + \frac{n}{4} = \frac{n}{2}$.

In expectation, 4 repeats are needed for $p_1$, and 4 repeats for $p_2$ (think: why?).