Approximation Algorithms 1: Vertex Cover and MAX-3SAT

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In computer science, there is a set of **NP-hard** problems such that

- nobody has found a polynomial-time algorithm for **any** of those problems;
- no polynomial-time algorithms can exist for **any** of those problems **unless** $\mathcal{P} = \mathcal{NP}$.

- $\mathcal{P}$ = the set of problems that can be solved in polynomial time on a **deterministic** Turing machine
- $\mathcal{NP}$ = the set of problems that can be solved in polynomial time on a **non-deterministic** Turing machine

Turing machines are formalized in CSCI3130 (Formal Languages and Automata Theory), and so is the notion of NP-hard.

Whether $\mathcal{P} = \mathcal{NP}$ is still unsolved to this day.

What can we do if a problem is NP-hard?

The rest of the course will focus on a principled approach for tackling NP-hard problems: **approximation**.

In many problems, even though an optimal solution may be expensive to find, we can find **near-optimal** solutions efficiently.

Next, we will see two examples: **vertex cover** and **MAX-3SAT**.

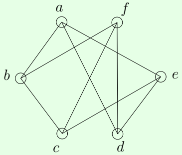
The Vertex Cover Problem

$G = (V, E)$ is a simple undirected graph.

A subset $S \subseteq V$ is a **vertex cover** of G if every edge $\{u, v\} \in E$ is incident to at least one vertex in S.

The V.C. Problem: Find a vertex cover of the smallest size.

Example:



An optimal solution is $\{a, f, c, e\}$.

The vertex cover problem is NP-hard.

- No one has found an algorithm solving the problem in time polynomial in $|V|$.
- Such algorithms cannot exist if $\mathcal{P} \neq \mathcal{NP}$.

Approximation Algorithms

$\mathcal{A}$ = an algorithm that, given any legal input $G = (V, E)$, returns a vertex cover of G.

OPT_G = the smallest size of all the vertex covers of G.

> $\mathcal{A}$ is a ρ-**approximate algorithm** for the vertex cover problem if, for any legal input $G = (V, E)$, $\mathcal{A}$ can return a vertex cover with size at most $\rho \cdot OPT_G$.

The value ρ is the **approximation ratio**.

We say that $\mathcal{A}$ achieves an approximation ratio of ρ.

Consider the following algorithm.

Input: $G = (V, E)$

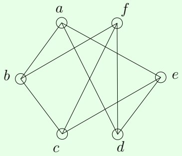
$S = \emptyset$
while E is not empty **do**
    pick an arbitrary edge $\{u, v\}$ in E
    add u, v to S
    remove from E all the edges of u and all the edges of v
return S

It is easy to show:

- S is a vertex cover of G;
- The algorithm runs in time polynomial to $|V|$ and $|E|$.

We will prove later that the algorithm is 2-approximate.

Example:



Suppose we start by picking edge $\{b, c\}$.
Then, $S = \{b, c\}$ and $E = \{\{a, e\}, \{a, d\}, \{d, e\}, \{d, f\}\}$.

Any edge in E can then be chosen. Suppose we pick $\{a, e\}$.
Then, $S = \{a, b, c, e\}$ and $E = \{\{d, f\}\}$.

Finally, pick $\{d, f\}$.
$S = \{a, b, c, d, e, f\}$ and $E = \emptyset$.

Theorem 1: The algorithm returns a set of at most $2 \cdot OPT_G$ vertices.

Let M be the set of edges picked.

Example: In the previous example, $M = \{\{b, c\}, \{a, e\}, \{d, f\}\}$.

Lemma 1: The edges in M do not share any vertices.

Proof: Suppose that M has edges e_1 and e_2 both incident to a vertex v. W.l.o.g., assume that e_1 was picked before e_2. After picking e_1, the algorithm deleted all the edges of v, because of which e_2 could not have been picked, giving a contradiction. □

Lemma 2: $|M| \le OPT_G$.

Proof: Any vertex cover must include at least one vertex of each edge in M. $|M| \le OPT$ follows from Lemma 1. □

Theorem 1 holds because the algorithm returns exactly $2|M|$ vertices.

The MAX-3SAT Problem

A **variable**: a boolean unknown x whose value is 0 or 1.
A **literal**: a variable x or its negation $\bar{x}$.
A **clause**: the OR of 3 literals with different variables.

S = a set of clauses
$\mathcal{X}$ = the set of variables appearing in at least one clause of S
A **truth assignment** of S: a function from $\mathcal{X}$ to $\{0, 1\}$.

A truth assignment f **satisfies** a clause in S if the clause evaluates to 1 under f.

The MAX-3SAT Problem: Let S be a set of n clauses. Find a truth assignment of S to maximize the number of clauses satisfied.

Example:

$$
\begin{aligned}
S = \{&x_1 \vee x_2 \vee x_3, \\
&x_1 \vee x_2 \vee \bar{x_3}, \\
&x_1 \vee \bar{x_2} \vee x_3, \\
&x_1 \vee \bar{x_2} \vee \bar{x_3}, \\
&\bar{x_1} \vee x_3 \vee x_4, \\
&\bar{x_1} \vee x_3 \vee \bar{x_4}, \\
&\bar{x_1} \vee \bar{x_3} \vee x_4. \\
&\bar{x_1} \vee \bar{x_3} \vee \bar{x_4}\}.
\end{aligned}
$$

$n = 8$ and $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$.

The truth assignment $x_1 = x_2 = x_3 = x_4 = 1$ satisfies 7 clauses. It is impossible to satisfy 8.

The MAX-3SAT problem is NP-hard.

- No one has found an algorithm solving the problem in time polynomial in n.
- Such algorithms cannot exist if $\mathcal{P} \neq \mathcal{NP}$.

Approximation Algorithms

$\mathcal{A}$ = an algorithm that, given any legal input S, returns a truth assignment of S.

OPT_S = the largest number of clauses that a truth assignment of S can satisfy.

Z_S = the number of clauses satisfied by the truth assignment $\mathcal{A}$ returns.

- Z_S is a random variable if $\mathcal{A}$ is randomized.

> $\mathcal{A}$ is a **randomized ρ-approximate algorithm** for MAX-3SAT if $\boldsymbol{E}[Z_S] \geq \rho \cdot OPT_S$ holds for any legal input S.

The value ρ is the **approximation ratio**.
We also say that $\mathcal{A}$ achieves an approximation ratio of ρ in expectation.

Consider the following algorithm.

Input: a set S of clauses with variable set $\mathcal{X}$

for each variable $x \in \mathcal{X}$ **do**
    toss a fair coin
    if the coin comes up heads **then** $x \leftarrow 1$
    else $x \leftarrow 0$

It is clear that the algorithm runs in $O(n)$ time.
Next, we show that the algorithm achieves an approximation ratio $7/8$ in expectation.

Theorem 2: The algorithm produces a truth assignment that satisfies $\frac{7}{8}n$ clauses in expectation.

Proof: It suffices to show that each clause is satisfied with probability $7/8$. W.l.o.g., suppose that the clause is $x_1 \vee x_2 \vee x_3$. The clause is 0 if and only if x_1, x_2, and x_3 are all 0. The probability for $x_1 = x_2 = x_3 = 0$ is $1/8$. $\square$

Think: What about a clause like $x_1 \vee x_2 \vee \bar{x}_3$?