

Review: Single Source Shortest Paths with Non-Negative Weights

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We will now commence our discussion on the **single source shortest path** (SSSP) problem. This lecture will start with **Dijkstra's algorithm**, which should have been covered in CSCI2100.

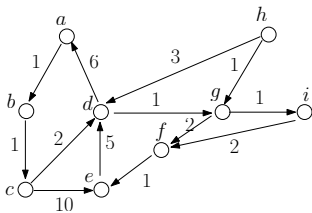
Weighted Graphs

Let $G = (V, E)$ be a simple directed graph.

Let w be a function that maps each edge $e \in E$ to a **non-negative** integer value $w(e)$, which we call the **weight** of e .

G and w together define a **weighted simple directed graph**.

Example



The integer on each edge indicates its weight. For example, $w(d, g) = 1$, $w(g, f) = 2$, and $w(c, e) = 10$.

Shortest Path

Consider a path in G : $(v_1, v_2), (v_2, v_3), \dots, (v_\ell, v_{\ell+1})$, for some integer $\ell \geq 1$. We define the path's **length** as

$$\sum_{i=1}^{\ell} w(v_i, v_{i+1}).$$

A **shortest path** from u to v has the minimum length among all the paths from u to v . Denote by $spdist(u, v)$ the length of a shortest path from u to v .

If v is unreachable from u , $spdist(u, v) = \infty$.

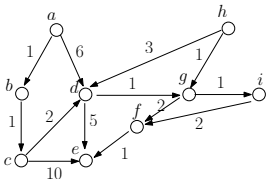
Single Source Shortest Path (SSSP) with Non-Negative Weights

Let $G = (V, E)$ be a simple directed graph, where function w maps every edge of E to a non-negative value. Given a **source vertex** s in V , we want to find a shortest path from s to t for **every** vertex $t \in V$ reachable from s .

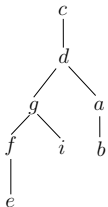
The output is a **shortest path tree** T :

- The vertex set of T is V .
- The root of T is s .
- For each node $u \in V$, the root-to- u path of T is a shortest path from s to u in G .

Example



A shortest path tree for source vertex c :



Edge Relaxation

For every vertex $v \in V$, we will — at all times — maintain a value $dist(v)$ equal to the shortest path length from s to v **found so far**.

Relaxing an edge (u, v) means:

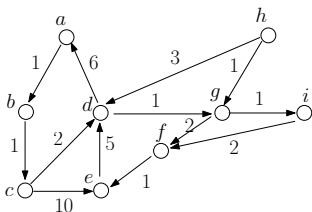
- If $dist(v) < dist(u) + w(u, v)$, do nothing;
- Otherwise, reduce $dist(v)$ to $dist(u) + w(u, v)$.

Dijkstra's Algorithm

- 1 Set $parent(v) \leftarrow \text{nil}$ for all vertices $v \in V$
- 2 Set $dist(s) \leftarrow 0$ and $dist(v) \leftarrow \infty$ for each vertex $v \in V \setminus \{s\}$
- 3 Set $S \leftarrow V$
- 4 Repeat the following until S is empty:
 - Remove from S the vertex u with the **smallest** $dist(u)$.
 - Relax every outgoing edge (u, v) of u .
If $dist(v)$ drops after the relaxation, set $parent(v) \leftarrow u$.

Example

Suppose that the source vertex is c .

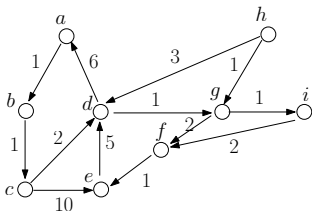


vertex v	$dist(v)$	$parent(v)$
a	∞	nil
b	∞	nil
c	0	nil
d	∞	nil
e	∞	nil
f	∞	nil
g	∞	nil
h	∞	nil
i	∞	nil

$$S = \{a, b, c, d, e, f, g, h, i\}.$$

Example

Relax the out-going edges of c .

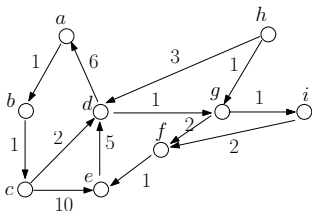


vertex v	$dist(v)$	$parent(v)$
a	∞	nil
b	∞	nil
c	0	nil
d	2	c
e	10	c
f	∞	nil
g	∞	nil
h	∞	nil
i	∞	nil

$$S = \{a, b, d, e, f, g, h, i\}.$$

Example

Relax the out-going edges of d .

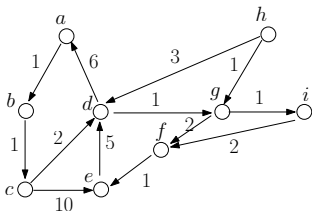


vertex v	$dist(v)$	$parent(v)$
a	8	d
b	∞	nil
c	0	nil
d	2	c
e	10	c
f	∞	nil
g	3	d
h	∞	nil
i	∞	nil

$$S = \{a, b, e, f, g, h, i\}.$$

Example

Relax the out-going edges of g .

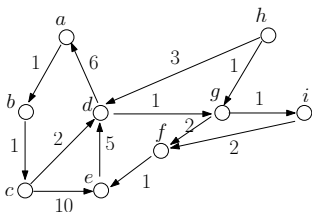


vertex v	$dist(v)$	$parent(v)$
a	8	d
b	∞	nil
c	0	nil
d	2	c
e	10	c
f	5	g
g	3	d
h	∞	nil
i	4	g

$$S = \{a, b, e, f, h, i\}.$$

Example

Relax the out-going edges of i .

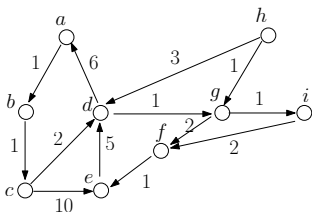


vertex v	$dist(v)$	$parent(v)$
a	8	d
b	∞	nil
c	0	nil
d	2	c
e	10	c
f	5	g
g	3	d
h	∞	nil
i	4	g

$$S = \{a, b, e, f, h\}.$$

Example

Relax the out-going edges of f .

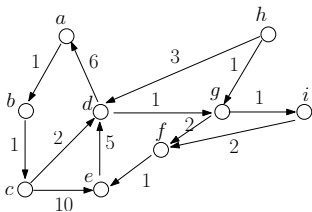


vertex v	$dist(v)$	$parent(v)$
a	8	d
b	∞	nil
c	0	nil
d	2	c
e	6	f
f	5	g
g	3	d
h	∞	nil
i	4	g

$$S = \{a, b, e, h\}.$$

Example

Relax the out-going edges of e .

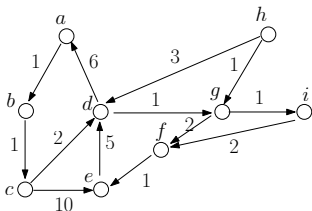


vertex v	$dist(v)$	$parent(v)$
a	8	d
b	∞	nil
c	0	nil
d	2	c
e	6	f
f	5	g
g	3	d
h	∞	nil
i	4	g

$$S = \{a, b, h\}.$$

Example

Relax the out-going edges of a .

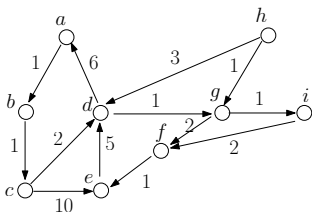


vertex v	$dist(v)$	$parent(v)$
a	8	d
b	9	a
c	0	nil
d	2	c
e	6	f
f	5	g
g	3	d
h	∞	nil
i	4	g

$S = \{b, h\}$.

Example

Relax the out-going edges of b .

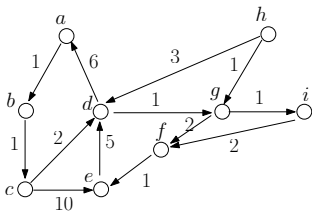


vertex v	$dist(v)$	$parent(v)$
a	8	d
b	9	a
c	0	nil
d	2	c
e	6	f
f	5	g
g	3	d
h	∞	nil
i	4	g

$$S = \{h\}.$$

Example

Relax the out-going edges of h .



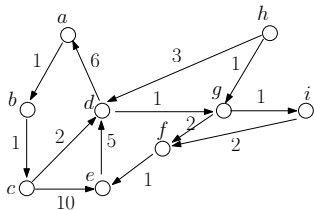
vertex v	$dist(v)$	$parent(v)$
a	8	d
b	9	a
c	0	nil
d	2	c
e	6	f
f	5	g
g	3	d
h	∞	nil
i	4	g

$S = \{\}$.

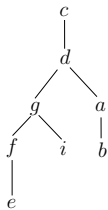
All the shortest path distances are now final.

Constructing the Shortest Path Tree

For every vertex v , if $u = \text{parent}(v)$ is not nil, then make v a child of u .



vertex v	$\text{parent}(v)$
a	d
b	a
c	nil
d	c
e	f
f	g
g	d
h	nil
i	g



You should be able to implement Dijkstra's algorithm to make sure that it runs in $O((|V| + |E|) \cdot \log |V|)$ time.

- Using advanced (graduate-level) data structures, we can reduce the time to $O(|V| \log |V| + |E|)$.

Dijkstra's algorithm does **not** work if edges can take negative weights.