ENGG1410F Tutorial
Quadratic Forms

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Symmetric matrices have many important applications. Today we will see one of them: determining whether a quadratic expression is positive definite.

Consider the expression:

$$x_1^2 + x_2^2 - x_3^2 - 2x_1x_2 + 2x_1x_3 + 2x_2x_3 \tag{1}$$

Clearly, if $x_1 = x_2 = x_3 = 0$, then the above expression is 0. We ask the question:

> Is it true that the expressive is always **positive** for any $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \neq \mathbf{0}$?

If so, then the expression is positive definite.

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The same question can be asked about any quadratic expressions, e.g.:

$$3x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3$$

We can convert

$$x_1^2 + x_2^2 - x_3^2 - 2x_1x_2 + 2x_1x_3 + 2x_2x_3$$

into the following "neat form":

$$\frac{1}{3}(x_1 + x_2 + x_3)^2 - 2\left(-\sqrt{\frac{1}{6}}x_1 - \sqrt{\frac{1}{6}}x_2 + \sqrt{\frac{2}{3}}x_3\right)^2 + (-x_1 - x_2)^2.$$

We now know that the original expression is not positive definite, e.g., the solution $\{x_1, x_2, x_3\}$ to

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ -\sqrt{\frac{1}{6}}x_1 - \sqrt{\frac{1}{6}}x_2 + \sqrt{\frac{2}{3}}x_3 &= 1 \\ -x_1 - x_2 &= 0 \end{aligned}$$

makes the expression negative.

Similarly, we can convert

$$3x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3$$

into the following "neat form":

$$\frac{1}{6}(x_1 + 2x_2 + x_3)^2 + \frac{3}{2}(-x_1 + x_3)^2 + \frac{4}{3}(x_1 - x_2 + x_3)^2.$$

We now know that the original expression is positive definite.

But here is the question:

> How to identify the above "neat" forms so that we can easily determine positive definiteness?

Next, we will give a systematic technique to do so, by resorting to a symmetric matrix.

First of all, observe:

$$
\begin{aligned}
& [x_1, x_2, x_3] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
= \quad & a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{21}x_2x_1 + \\
& a_{22}x_2^2 + a_{23}x_2x_3 + a_{31}x_3x_1 + a_{32}x_3x_2 + a_{33}x_3^2.
\end{aligned}
$$

The matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is called the coefficient matrix of the quadratic expression.

> For our technique to work, we require that the coefficient matrix should be symmetric!

Fortunately, every quadratic expression admits a symmetric coefficient matrix; see the next slide for an example.

$$\begin{aligned} & x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3 \\ = \; & x_1^2 - 2x_1x_2 + 4x_1x_3 - 2x_2x_1 + 2x_2^2 + 4x_3x_1 - 7x_3^2 \\ = \; & [x_1, x_2, x_3] \begin{bmatrix} 1 & -2 & 4 \\ -2 & 2 & 0 \\ 4 & 0 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \end{aligned}$$

Let $\boldsymbol{A} = \begin{bmatrix} 1 & -2 & 4 \\ -2 & 2 & 0 \\ 4 & 0 & -7 \end{bmatrix}$.

$\boldsymbol{A}$ has eigenvalues $\lambda_1 = 1, \lambda_2 = -2, \lambda_3 = 2$.

As $\boldsymbol{A}$ is symmetric, we know that it can be diagonalized into $\boldsymbol{QBQ}^{-1}$ where $\boldsymbol{Q}$ is an orthogonal matrix, and $\boldsymbol{B} = diag[1, -2, 2]$. With this we obtain:

$$\begin{aligned}
[x_1, x_2, x_3]\boldsymbol{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= [x_1, x_2, x_3]\boldsymbol{QBQ}^{-1}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
\text{(by } \boldsymbol{Q}^{-1} = \boldsymbol{Q}^T\text{)} &= [x_1, x_2, x_3]\boldsymbol{QBQ}^T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&= \Big([x_1, x_2, x_3]\boldsymbol{Q}\Big)\boldsymbol{B}\Big([x_1, x_2, x_3]\boldsymbol{Q}\Big)^T
\end{aligned}$$

Let $[y_1, y_2, y_3] = [x_1, x_2, x_3]\boldsymbol{Q}$, then we can write the above as

$$[y_1, y_2, y_3]\boldsymbol{B}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1^2 - 2y_2^2 + 2y_3^2.$$

As we will see, this is the "neat form" we are looking for.

Recall that for the expression $x_1^2 + x_2^2 - x_3^2 - 2x_1x_2 + 2x_1x_3 + 2x_2x_3$,
$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ which equals $\boldsymbol{Q} diag[1, -2, 2] \boldsymbol{Q}^{-1}$ where

$$\boldsymbol{Q} = \begin{bmatrix} 1/\sqrt{3} & -\sqrt{1/6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -\sqrt{1/6} & -1\sqrt{2} \\ 1/\sqrt{3} & \sqrt{2/3} & 0 \end{bmatrix}$$

Accordingly:

$$\begin{aligned} y_1 &= (x_1 + x_2 + x_3)/\sqrt{3} \\ y_2 &= -\sqrt{1/6} \cdot x_1 - \sqrt{1/6} \cdot x_2 + \sqrt{2/3} \cdot x_3 \\ y_3 &= -(x_1 + x_2)/\sqrt{2} \end{aligned}$$

This gives precisely the neat form in Slide 4.

Next, let us apply the technique to prove

$$3x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3$$

is positive definite.

First, write:

$$\begin{aligned} & 3x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3 \\ = \quad & [x_1, x_2, x_3] \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \end{aligned}$$

Let $\boldsymbol{A} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$.

Diagonalize $\boldsymbol{A}$ into $\boldsymbol{QBQ}^{-1}$ where

$$\begin{aligned} \boldsymbol{Q} &= \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \\ \boldsymbol{B} &= diag[1, 3, 4] \end{aligned}$$

Accordingly:

$$\begin{aligned} y_1 &= (x_1 + 2x_2 + x_3)/\sqrt{6} \\ y_2 &= (-x_1 + x_3)/\sqrt{2} \\ y_3 &= (x_1 - x_2 + x_3)/\sqrt{3} \end{aligned}$$

and

$$\boldsymbol{A} = y_1^2 + 3y_2^2 + 4y_3^2.$$

This gives precisely the neat form in Slide 5.

The above technique can be summarized into the following algorithm for deciding whether a quadratic expression is positive definite:

1. Obtain the symmetric coefficient matrix $\boldsymbol{A}$ of the expression.
2. Obtain all the eigenvalues of $\boldsymbol{A}$.
3. If all eigenvalues are positive, then the original expression is positive definite.
4. Otherwise, not positive definite.

Remark 1: Although we have illustrated the algorithm for $n = 3$ variables, the technique can be generalized in a straightforward manner to any n (in any case $\boldsymbol{A}$ is an $n \times n$ matrix).

Remark 2: A symmetric $n \times n$ matrix $\boldsymbol{A}$ is said to be positive definite if $\boldsymbol{x}^T \boldsymbol{A}\boldsymbol{x} > 0$ for any $n \times 1$ vector $\boldsymbol{x}$. Our argument earlier showed that $\boldsymbol{A}$ is positive definite if and only if all its eigenvalues are positive.