ENGG1410-F Tutorial 6

Jianwen Zhao

Department of Computer Science and Engineering
The Chinese University of Hong Kong

Problem 1. Matrix Diagonalization

Diagonalize the following matrix:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

Solution

The 2×2 matrix $\boldsymbol{A}$ has two distinct eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 5$, which means it is diagonalizable.

We then obtain an arbitrary eigenvector $\boldsymbol{v}_1$ of λ_1 and also an arbitrary eigenvector $\boldsymbol{v}_2$ of λ_2, say

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ \ \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Next, apply the diagonalization method we discussed in class, form:

$$\boldsymbol{Q} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

by using $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ as the first and second column respectively.

Solution–cont.

$\boldsymbol{Q}$ has the inverse

$$\boldsymbol{Q}^{-1} = \begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{bmatrix}$$

We thus obtain the following diagonalization of $\boldsymbol{A}$:

$$\boldsymbol{A} = \boldsymbol{Q}\ diag[-1, 5]\ \boldsymbol{Q}^{-1}$$

Problem 2. Matrix Power

Consider again the matrix $\boldsymbol{A}$ in Problem 1, i.e,.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

Calculate $\boldsymbol{A}^t$ for any integer $t \geq 1$.

Solution

We already know that

$$\boldsymbol{A} = \boldsymbol{Q}\ diag[-1, 5]\ \boldsymbol{Q}^{-1}$$

Hence,

$$\begin{aligned}
\boldsymbol{A}^t &= \boldsymbol{Q}\ diag[(-1)^t, 5^t]\ \boldsymbol{Q}^{-1} \\
&= \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} (-1)^t & 0 \\ 0 & 5^t \end{bmatrix} \begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{bmatrix} \\
&= \begin{bmatrix} (5^t + 2 \times (-1)^t)/3 & (5^t + (-1)^{t+1})/3 \\ (2 \times 5^t + 2 \times (-1)^{t+1})/3 & (2 \times 5^t + (-1)^{t+2})/3 \end{bmatrix}
\end{aligned}$$

Problem 3. Matrix Diagonalization

Diagonalize the following matrix:

$$\boldsymbol{A} = \begin{bmatrix} 4 & -3 & -3 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{bmatrix}$$

Solution

$\boldsymbol{A}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$. $EigenSpace(\lambda_1)$ includes all $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ satisfying $x_1 = u + v, x_2 = u, x_3 = v$ for any $u, v \in \mathbb{R}$.

The vector space $EigenSpace(\lambda_1)$ has dimension 2 with a basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ where $\boldsymbol{v}_1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$ (given by $u = 1, v = 0$) and $\boldsymbol{v}_2 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$ (given by $u = 0, v = 1$).

Similarly, $EigenSpace(\lambda_2)$ includes all $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ satisfying $x_1 = x_2 = -3u$ and $x_3 = u$ for any $u \in \mathbb{R}$.

The vector space $EigenSpace(\lambda_2)$ has dimension 1 with a basis $\{\boldsymbol{v}_3\}$ where $\boldsymbol{v}_3 = \begin{bmatrix} -3 & -3 & 1 \end{bmatrix}^T$ (given by $u = 1$).

Solution-cont.

So far, we have obtained three linearly independent eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ of $\boldsymbol{A}$. We then construct

$$\boldsymbol{Q} = \begin{bmatrix} 1 & 1 & -3 \\ 1 & 0 & -3 \\ 0 & 1 & 1 \end{bmatrix}$$

and $\boldsymbol{Q}$ has the inverse

$$\boldsymbol{Q}^{-1} = \begin{bmatrix} -3 & 4 & 3 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

We thus obtain the following diagonalization of $\boldsymbol{A}$:

$$\boldsymbol{A} = \boldsymbol{Q}\; diag[1, 1, 2]\; \boldsymbol{Q}^{-1}$$

Problem 4. Matrix Similarity

Suppose that matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are similar to each other, namely, there exists $\boldsymbol{P}$ such that $\boldsymbol{A} = \boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P}$.

Prove: if $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$ under eigenvalue λ, then $\boldsymbol{P}\boldsymbol{x}$ is an eigenvector of $\boldsymbol{B}$ under eigenvalue λ.

Problem 5. Matrix Trace

Definition. The trace of an $n \times n$ square matrix $\boldsymbol{A}$, denoted by $tr(A)$, is defined to be the sum of the elements on the main diagonal of $\boldsymbol{A}$, i.e., $tr(\boldsymbol{A}) = \sum_{i=1}^{n} a_{ii}$.

For example, if

$$\boldsymbol{A} = \begin{bmatrix} 4 & -3 & -3 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{bmatrix}$$

then $tr(\boldsymbol{A}) = 4 + (-2) + 2 = 4$.

Prove: $tr(\boldsymbol{AB}) = tr(\boldsymbol{BA})$, where $\boldsymbol{A}$ is an $m \times n$ matrix and $\boldsymbol{B}$ is an $n \times m$ matrix.

Solution

Proof. Denote by a_{ij} the element of $\boldsymbol{A}$ at i-th row and j-th column, b_{ji} the element of $\boldsymbol{B}$ at j-th row and i-th column, where $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$. Then

$$(\boldsymbol{AB})_{ii} = a_{i1}b_{1i} + a_{i2}b_{2i} + \cdots + a_{in}b_{ni} = \sum_{j=1}^{n} a_{ij}b_{ji}$$

Similarly,

$$(\boldsymbol{BA})_{jj} = b_{j1}a_{1j} + b_{j2}a_{2j} + \cdots + b_{jm}a_{mj} = \sum_{i=1}^{m} b_{ji}a_{ij}$$

Hence

$$tr(\boldsymbol{AB}) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{m} b_{ji}a_{ij} = tr(\boldsymbol{BA})$$

□

Problem 6. Traces & Eigenvalues & Determinants

Suppose $\boldsymbol{A}$ is an $n \times n$ diagonalizable matrix, namely, there exists $\boldsymbol{Q}$ such that $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{B}\boldsymbol{Q}^{-1}$, and $\boldsymbol{B}$ is a diagonal matrix. Denote by $\lambda_1, \lambda_2, \cdots, \lambda_n$ the n eigenvalues of $\boldsymbol{A}$.

Prove: (1) $tr(\boldsymbol{A}) = \sum_{i=1}^{n} \lambda_i$, (2) $det(\boldsymbol{A}) = \Pi_{i=1}^{n} \lambda_i$.

Solution

Proof.

(1)

$$\begin{aligned} tr(\boldsymbol{A}) &= tr(\boldsymbol{Q}\boldsymbol{B}\boldsymbol{Q}^{-1}) \\ &= tr(\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{Q}) \\ &= tr(\boldsymbol{B}) \\ &= \sum_{i=1}^{n} \lambda_i \end{aligned}$$

Where the second equality used the fact that $tr(\boldsymbol{AB}) = tr(\boldsymbol{BA})$ and the last equality used the facts (i) $\boldsymbol{A}$ and $\boldsymbol{B}$ have exactly the same eigenvalues due to their similarity, and (ii) the eigenvalues of a diagonal matrix are simply its diagonal elements.

Solution-cont.

(2)

$$
\begin{aligned}
det(\boldsymbol{A}) &= det(\boldsymbol{Q}\boldsymbol{B}\boldsymbol{Q}^{-1}) \\
&= det(\boldsymbol{Q}) \cdot det(\boldsymbol{B}) \cdot det(\boldsymbol{Q}^{-1}) \\
&= det(\boldsymbol{B}) \cdot det(\boldsymbol{Q}) \cdot det(\boldsymbol{Q}^{-1}) \\
&= det(\boldsymbol{B}) \cdot det(\boldsymbol{Q}\boldsymbol{Q}^{-1}) \\
&= det(\boldsymbol{B}) \\
&= \Pi_{i=1}^{n} \lambda_i
\end{aligned}
$$

Where the last equality used the facts (i) $\boldsymbol{A}$ and $\boldsymbol{B}$ have exactly the same eigenvalues due to their similarity, and (ii) the eigenvalues of a diagonal matrix are simply its diagonal elements. □

In fact, the conclusion of this problem is true in general, regardless of whether $\boldsymbol{A}$ is diagonalizable.

> For any $n \times n$ square matrix $\boldsymbol{A}$, if its n eigenvalues are $\lambda_1, \lambda_2, \cdots, \lambda_n$, then $tr(\boldsymbol{A}) = \sum_{i=1}^{n} \lambda_i$ and $det(\boldsymbol{A}) = \Pi_{i=1}^{n} \lambda_i$.

The proof is not difficult but a little tedious, students who are interested may refer to the proof at the following link:

https://www.adelaide.edu.au/mathslearning/play/seminars/evalue-magic-tricks-handout.pdf