Exercises: Determinant

Problem 1. Calculate the determinant of the following matrix:

$$\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$$

Solution. We can do so by applying the definition of determinant. Specifically, expanding the matrix by the first row gives:

$$\begin{aligned}
\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} &= a\begin{vmatrix} a & b \\ c & a \end{vmatrix} - b\begin{vmatrix} c & b \\ b & a \end{vmatrix} + c\begin{vmatrix} c & a \\ b & c \end{vmatrix} \\
&= a^3 - abc - abc + b^3 + c^3 - abc \\
&= a^3 + b^3 + c^3 - 3abc.
\end{aligned}$$

Problem 2. Calculate the determinant of the following matrix:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Solution.

$$\begin{aligned}
\begin{vmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix} &= \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix} \\
&= -\begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} \\
&= -\begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} = -1.
\end{aligned}$$

Problem 3. Calculate the determinant of the following matrix:

$$\begin{bmatrix} 0 & 4 & -6 \\ 4 & 0 & 10 \\ -6 & 10 & 0 \end{bmatrix}$$

Solution.

$$\begin{aligned}
\begin{vmatrix} 0 & 4 & -6 \\ 4 & 0 & 10 \\ -6 & 10 & 0 \end{vmatrix} &= -\begin{vmatrix} 4 & 0 & 10 \\ 0 & 4 & -6 \\ -6 & 10 & 0 \end{vmatrix} \\
&= \begin{vmatrix} 4 & 0 & 10 \\ -6 & 10 & 0 \\ 0 & 4 & -6 \end{vmatrix} \\
&= \begin{vmatrix} 4 & 0 & 10 \\ 0 & 10 & 15 \\ 0 & 4 & -6 \end{vmatrix} \\
&= 4 \times \begin{vmatrix} 10 & 15 \\ 4 & -6 \end{vmatrix} \\
&= 4 \times (-60 - 60) = -480.
\end{aligned}$$

Problem 4. Suppose that $\boldsymbol{A}$ is an $n \times n$ matrix. Prove: $det(c\boldsymbol{A}) = c^n det(\boldsymbol{A})$.

Proof. Recall that every time we multiply a row of $\boldsymbol{A}$ by c, the determinant of the matrix increases by a factor of c. To obtain $c\boldsymbol{A}$, we need to multiple each of the n rows of $\boldsymbol{A}$ by c. □

Problem 5. Calculate

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{vmatrix}$$

Solution. Expanding the matrix by the first row, we get:

$$\begin{aligned}
&\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{vmatrix} \\
= \; & a_{11} \begin{vmatrix} a_{22} & 0 & 0 \\ a_{32} & 0 & 0 \\ a_{42} & 0 & 0 \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \\ a_{41} & 0 & 0 \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \\ a_{41} & a_{42} & 0 \end{vmatrix} - a_{14} \begin{vmatrix} a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \\ a_{41} & a_{42} & 0 \end{vmatrix} \\
= \; & 0.
\end{aligned}$$

Problem 6. Calculate

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 & 0 \\ a_{51} & a_{52} & 0 & 0 & 0 \end{vmatrix}$$

Solution. Expanding the matrix by the 1st row, we get:

$$\begin{aligned}
&\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 & 0 \\ a_{51} & a_{52} & 0 & 0 & 0 \end{vmatrix} \\
= \; & a_{11} \begin{vmatrix} a_{22} & a_{23} & a_{24} & a_{25} \\ a_{32} & 0 & 0 & 0 \\ a_{42} & 0 & 0 & 0 \\ a_{52} & 0 & 0 & 0 \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} & a_{24} & a_{25} \\ a_{31} & 0 & 0 & 0 \\ a_{41} & 0 & 0 & 0 \\ a_{51} & 0 & 0 & 0 \end{vmatrix} + \\
& a_{13} \begin{vmatrix} a_{21} & a_{22} & a_{24} & a_{25} \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \\ a_{51} & a_{52} & 0 & 0 \end{vmatrix} - a_{14} \begin{vmatrix} a_{21} & a_{22} & a_{23} & a_{25} \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \\ a_{51} & a_{52} & 0 & 0 \end{vmatrix} + \\
& a_{15} \begin{vmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \\ a_{51} & a_{52} & 0 & 0 \end{vmatrix} \\
\text{(by the result of Problem 5)} \; = \; & 0.
\end{aligned}$$

Problem 7. Let $\boldsymbol{A}$ be an $n \times n$ matrix. Prove:

- If we switch two columns of $\boldsymbol{A}$, $det(\boldsymbol{A})$ gets multiplied by -1.
- If we multiply a column of $\boldsymbol{A}$ by a non-zero value α, $det(\boldsymbol{A})$ gets multiplied by α.
- Let $\boldsymbol{c}_i$ and $\boldsymbol{c}_j$ be two different columns of $\boldsymbol{A}$. If we replace $\boldsymbol{c}_i$ by $\boldsymbol{c}_i + \alpha \boldsymbol{c}_j$, $det(\boldsymbol{A})$ remains the same.

Proof. Remember that $det(\boldsymbol{A}) = det(\boldsymbol{A}^T)$. The above statements are correct because the described operations are elementary row operations on $\boldsymbol{A}^T$. □

Problem 8. Calculate

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1-a & 1 & 1 \\ 1 & 1 & 1+b & 1 \\ 1 & 1 & 1 & 1-b \end{vmatrix}.$$

Solution.

$$\begin{aligned}
\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1-a & 1 & 1 \\ 1 & 1 & 1+b & 1 \\ 1 & 1 & 1 & 1-b \end{vmatrix}
&= \begin{vmatrix} a & a & 0 & 0 \\ 1 & 1-a & 1 & 1 \\ 0 & a & b & 0 \\ 0 & a & 0 & -b \end{vmatrix} \\
&= \begin{vmatrix} a & a & 0 & 0 \\ 1 & 1-a & 1 & 1 \\ 0 & a & b & 0 \\ 0 & 0 & -b & -b \end{vmatrix} \\
&= ab\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1-a & 1 & 1 \\ 0 & a & b & 0 \\ 0 & 0 & -1 & -1 \end{vmatrix} \\
&= ab\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & -a & 1 & 1 \\ 0 & a & b & 0 \\ 0 & 0 & -1 & -1 \end{vmatrix} \\
&= ab\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & -a & 1 & 1 \\ 0 & 0 & b+1 & 1 \\ 0 & 0 & -1 & -1 \end{vmatrix} \\
&= -ab\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & -a & 1 & 1 \\ 0 & 0 & 1 & b+1 \\ 0 & 0 & -1 & -1 \end{vmatrix} \\
&= -ab\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & -a & 1 & 1 \\ 0 & 0 & 1 & b+1 \\ 0 & 0 & 0 & b \end{vmatrix} \\
&= a^2b^2.
\end{aligned}$$