Exercises: Matrix Basic Operations and Gauss Elimination

Problem 1. Let $\boldsymbol{A}$ be a square $n \times n$ matrix, and $\boldsymbol{I}$ an identity $n \times n$ matrix. Prove $\boldsymbol{AI} = \boldsymbol{A}$, and $\boldsymbol{IA} = \boldsymbol{A}$.

Proof: We will prove only $\boldsymbol{AI} = \boldsymbol{A}$ because the argument for $\boldsymbol{IA} = \boldsymbol{A}$ is similar. Denote by $\boldsymbol{B}$ the product of $\boldsymbol{AI}$. Let $\boldsymbol{A} = [a_{ij}]$, $\boldsymbol{B} = [b_{ij}]$, and $\boldsymbol{I} = [e_{ij}]$. We have:

$$b_{ij} = \sum_{k=1}^{n} a_{ik} e_{kj}$$

As $\boldsymbol{I}$ is an identity matrix, we know that $e_{kj} = 1$ if $k = j$, while $e_{kj} = 0$ if $k \neq j$. Therefore, the right hand side of the above equals a_{ij}. □

Problem 2. Calculate $\boldsymbol{AB}$, $\boldsymbol{BA}$, and $\boldsymbol{A}^T\boldsymbol{B}^T$, where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & 1 \\ -1 & -2 & 1 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

Solution.

$$\boldsymbol{AB} = \begin{bmatrix} 5 & -1 & 1 \\ 4 & 1 & 2 \\ -4 & -2 & -1 \end{bmatrix}, \boldsymbol{BA} = \begin{bmatrix} 3 & 4 & 6 \\ 1 & 3 & 2 \\ -2 & 0 & -1 \end{bmatrix}, \boldsymbol{A}^T\boldsymbol{B}^T = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 3 & 0 \\ 6 & 2 & -1 \end{bmatrix}.$$

Problem 3. $\boldsymbol{A}, \boldsymbol{B}$, and $\boldsymbol{C}$ are $m \times n$, $n \times p$, and $p \times q$ matrices. Prove: $(\boldsymbol{ABC})^T = \boldsymbol{C}^T\boldsymbol{B}^T\boldsymbol{A}^T$.

Proof.

$$\begin{aligned} (\boldsymbol{ABC})^T &= \boldsymbol{C}^T(\boldsymbol{AB})^T \\ &= \boldsymbol{C}^T\boldsymbol{B}^T\boldsymbol{A}^T. \end{aligned}$$

□

Problem 4. What is $\boldsymbol{A}^T$ if $\boldsymbol{A}$ is (i) symmetric, and (ii) anti-symmetric?

Solution. $\boldsymbol{A}$ is symmetric if and only if $\boldsymbol{A} = \boldsymbol{A}^T$. Also, $\boldsymbol{A}^T$ is anti-symmetric if and only if $\boldsymbol{A} = -\boldsymbol{A}^T$.

Problem 5. $\boldsymbol{A}$ and $\boldsymbol{B}$ are both $n \times n$ symmetric matrices. Prove: AB is symmetric if and only if $AB = BA$.

Proof. AB is symmetric if and only if $AB = (AB)^T = B^TA^T = BA$. □

Problem 6. Consider the following recurrence for $i \geq 1$:

$$\boldsymbol{x_{i+1}} = \boldsymbol{Ax_i}$$

where $\boldsymbol{A}$ is an 3×3 matrix, and $\boldsymbol{x_i}$ and $\boldsymbol{x_{i+1}}$ are 3×1 matrices. Knowing:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \text{ and } \boldsymbol{x_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

what is the value of $\boldsymbol{x_3}$?

Solution 1.

$$\begin{aligned} \boldsymbol{x_2} &= \boldsymbol{A}\boldsymbol{x_1} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}. \end{aligned}$$

$$\begin{aligned} \boldsymbol{x_3} &= \boldsymbol{A}\boldsymbol{x_2} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}. \end{aligned}$$

Solution 2.

$$\begin{aligned} \boldsymbol{x_3} &= \boldsymbol{A}\boldsymbol{x_2} \\ &= \boldsymbol{A}^2\boldsymbol{x_1} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}. \end{aligned}$$

Problem 7. Convert the following matrix into row echelon form with elementary row operations:

$$\begin{bmatrix} 0 & 3 & 1 & 1 \\ 0 & 0 & 5 & 5 \\ 1 & -1 & 3 & 3 \\ 3 & 3 & -7 & -7 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 0 & 0 & 5 & 5 \\ 0 & 3 & 1 & 1 \\ 1 & -1 & 3 & 3 \\ 3 & 3 & -7 & -7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 3 & 3 \\ 3 & 3 & -7 & -7 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 5 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 3 & 3 \\ 0 & 6 & -16 & -16 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 5 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 3 & 3 \\ 0 & 6 & -16 & -16 \\ 0 & 0 & 9 & 9 \\ 0 & 0 & 5 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 3 & 3 \\ 0 & 6 & -16 & -16 \\ 0 & 0 & 9 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 8. Solve the following linear system with Gauss Elimination

$$\begin{aligned} 4y + 3z &= 8 \\ 2x - z &= -2 \\ x + 2z &= 5. \end{aligned}$$

Solution. First, obtain the augmented matrix:

$$\begin{bmatrix} 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & -2 \\ 1 & 0 & 2 & 5 \end{bmatrix}$$

Next, convert the matrix into row echelon form:

$$\begin{bmatrix} 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & -2 \\ 1 & 0 & 2 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & -1 & -2 \\ 1 & 0 & 2 & 5 \\ 0 & 4 & 3 & 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & -1 & -2 \\ 0 & 0 & 5/2 & 6 \\ 0 & 4 & 3 & 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & -1 & -2 \\ 0 & 4 & 3 & 8 \\ 0 & 0 & 5/2 & 6 \end{bmatrix}$$

Now apply back substitution to obtain the solution of x, y, z. Specifically, from

$$(5/2)z = 6$$

we get $z = 12/5$. From

$$4y + 3z = 8$$

we get $y = 1/5$. From

$$2x - z = -2$$

we get $x = 1/5$.

Problem 9. Decide if the following linear system is consistent.

$$\begin{aligned} 4y + 3z &= 8 \\ 2x - z &= -2 \\ x + 2y + z &= 3. \end{aligned}$$

If it is, give all the solutions to the system.

Solution. Augmented matrix:

$$\begin{bmatrix} 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & -2 \\ 1 & 2 & 1 & 3 \end{bmatrix}$$

Convert it to row echelon form:

$$\begin{bmatrix} 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & -2 \\ 1 & 2 & 1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & -1 & -2 \\ 1 & 2 & 1 & 3 \\ 0 & 4 & 3 & 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & -1 & -2 \\ 0 & 2 & 3/2 & 4 \\ 0 & 4 & 3 & 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & -1 & -2 \\ 0 & 2 & 3/2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding linear system:

$$\begin{aligned} 2x - z &= -2 \\ 2y + (3/2)z &= 4 \end{aligned}$$

It is thus clear that the system has infinitely many solutions. To find them all, introduce a parameter t. Then we know that any $x = (t/2) - 1, y = 2 - 3t/4$, and $z = t$ is a solution of the original system.